Euler Systems Lecture in Lausanne:
A tamely ramified variant of
a conjecture of Perrin-Riou

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Abstract

This is a (somewhat expanded) set of notes for a lecture given in Lausanne in August 2017. It is not intended for publication, and the reader should not be misled by the tex layout into treating it as anything other than an informal transcript, rife with assertions that are vague at best, and possibly incomplete and inaccurate. In particular, it should not be cited. This write-up is based on numerous exchanges with Michael Harris, Alan Lauder, Victor Rotger and Akshay Venkatesh. While the ideas it contains should also be credited to them, the mistakes are to be blamed on the author alone.

1 Mazur-Tate conjectures and Euler systems

In the summer of 1986, Barry Mazur, John Tate and Dick Gross ran a seminar at Harvard on “tamely ramified” refinements of the $p$-adic Birch and Swinnerton-Dyer and Gross-Stark conjectures. A year later, Kolyvagin’s remarkable proof of the BSD conjecture in analytic rank $\leq 1$ was announced, just as I was starting my graduate studies at Harvard. Dick Gross immediately saw that Kolyvagin’s ideas should be understood within the Mazur-Tate framework of “tamely ramified Iwasawa theory”, and asked me to flesh out this insight. The goal of
this lecture is to revisit the connections between Euler systems and the Mazur-Tate conjectures in the very special, but nonetheless illustrative, setting of Garrett-Rankin triple products. An important motivation for wanting to understand the general picture is its relevance to the conjectures of [V], [PV] on derived Hecke operators in the setting of $\text{GL}_2$, which Michael Harris will speak about this afternoon.

2 The $p$-adic setting

It often happens that a motive $M$ (more accurately: its $p$-adic étale realisation) arises as the specialisation of a $p$-adic family $\mathcal{M}$ varying over a $p$-adic parameter space $\mathcal{X}$. The key $p$-adic invariants attached to the specialisations $\mathcal{M}_x$ for suitable $x \in \mathcal{X}$ are then expected to inherit similar $p$-adic interpolation properties, notably:

1. The “algebraic parts” of the special values $L(\mathcal{M}_x, 0)$ of the $L$-functions attached to $\mathcal{M}_x$, for $x$ in a suitable subset $\mathcal{X}_{\text{cl}} \subset \mathcal{X}$. The interpolating object, denoted $L(\mathcal{M}) \in \mathcal{O}_{\mathcal{X}}$, is called the $p$-adic $L$-function attached to the pair $(\mathcal{M}, \mathcal{X}_{\text{cl}})$, and the set $\mathcal{X}_{\text{cl}}$ is called its “range of classical interpolation”. The $p$-adic $L$-function $L(\mathcal{M})$ is uniquely determined if $\mathcal{X}_{\text{cl}}$ is suitably dense in $\mathcal{X}$, as is often the case.

2. The Bloch-Kato Selmer groups $H^1_f(\mathbb{Q}, \mathcal{M}_x)$ attached to $\mathcal{M}_x$, for $x \in \mathcal{X}_{\text{cl}}$, whose Fitting ideas as $\mathcal{O}_{\mathcal{X}}$-modules should be related to $L(\mathcal{M})$ via a generalised “Iwasawa main conjecture”.

3. Motivic extension classes $\kappa_x \in H^1_f(\mathbb{Q}, \mathcal{M}_x)$, for $x$ in a suitable subset $\mathcal{X}_{\text{geom}} \subset \mathcal{X}$. These classes are typically constructed by geometric means from special elements in (higher) Chow groups, and their images under (complex or $p$-adic) regulators account for the leading terms of $L(\mathcal{M}_x, 0)$ following the conjectures of Beilinson-Bloch. The subset $\mathcal{X}_{\text{geom}} \subset \mathcal{X}$ is called the “range of geometric interpolation” for the class $\kappa \in H^1(\mathbb{Q}, \mathcal{M})$ interpolating the $\{\kappa_x\}_{x \in \mathcal{X}_{\text{geom}}}$. This class is uniquely determined if $\mathcal{X}_{\text{geom}}$ is suitably dense in $\mathcal{X}$.

One expects that $\kappa$ determines the $p$-adic $L$-function $L(\mathcal{M})$, which can be recovered from the image of $\kappa$ under a “$p$-adic regulator map in families”. The specialised classes $\kappa_x$ for $x \in \mathcal{X}_{\text{cl}}$ enjoy a rich structure and can be used to bound (or at least, provide explicit annihilators for)
$H^1_f(Q, M_x)$, and its relation with $L(M_x) \sim L(M_x, 0)$ thus provides an eventual bridge between $L(M_x, 0)$ and $H^1_f(Q, M_x)$ suitable for tackling the main conjecture.

Two settings that have been explored most systematically so far are:

1. The “Iwasawa theoretic” setting where $X = \text{Spec}(\Lambda)$ and $\Lambda$ is the Iwasawa algebra attached to the Galois group $\Gamma$ of a $\mathbb{Z}_p$-extension. Points of $X$ correspond to continuous $p$-adic characters of $\Gamma$, and one can let $M$ be the family whose specialisation at $x$ is $M(x)$, the twist of a fixed $M$ by this character. Most closely related to the classical archimedean setting is the case where $\Gamma$ is the Galois group of the cyclotomic $\mathbb{Z}_p$-extension, where the variable $x$ is a direct $p$-adic avatar of the $s$-variable of the complex theory.

2. The “Hida theoretic” setting where $M$ is a more complicated family of irreducible Galois representations, in which the spread of Hodge-Tate weights could be varying for instance. The prototypical example is the family of Galois representations arising from a Hida family or more general Coleman family of automorphic representations.

Examples of classes $\kappa$ that fit neatly in the “Iwasawa theoretic setting” include:

1. the cyclotomic family arising from circular units, where $\Gamma$ is the Galois group of the cyclotomic $\mathbb{Z}_p$ extension of $\mathbb{Q}$ and $M = \mathbb{Q}_p$;

2. the family arising from elliptic units, where $\Gamma \simeq \mathbb{Z}_p^2$ is the Galois group of the two-variable $\mathbb{Z}_p$-extension of an imaginary quadratic field, and $M = \mathbb{Q}_p$;

3. the family arising from Heegner points, where $\Gamma$ is the Galois group of the anticyclotomic $\mathbb{Z}_p$-extension of an imaginary quadratic field, and $M$ is the Galois representation attached to a classical modular form;

4. the cyclotomic family arising from Kato’s Beilinson elements in $p$-adic families, where $M$ is again the Galois representation arising from a classical modular form;

5. the cyclotomic family of Beilinson-Flach elements, where $M = M_1 \otimes M_2$ is the tensor product of two Galois representations attached to classical modular forms;
6. the rich panoply of new cyclotomic examples being explored in
the ongoing work of Cornut, Jetchev, Kings, Lei, Lemma, Loeffler, Skinner, Zerbes, ... which are among the major themes of
this special semester.

3 Triple products and diagonal cycles

The simplest example of a family of classes $\kappa$ as above whose
underlying family of Galois representations fails to fit into the Iwasawa
theoretic framework arises when this Galois representation is the triple
tensor product of the $\Lambda$-adic representations attached to three Hida
families $f$, $g$ and $h$:

$$
\mathcal{M}_{fgh} := \mathcal{M}_f \otimes \mathcal{M}_g \otimes \mathcal{M}_h,
$$

where $\mathcal{M}_f$, $\mathcal{M}_g$ and $\mathcal{M}_h$ are the $\Lambda$-adic representations attached to $f$, $g$, and $h$ respectively. For notational simplicity, we assume that $\mathcal{M}_f$, $\mathcal{M}_g$, and $\mathcal{M}_h$ are free of rank two over the Iwasawa algebra $\Lambda$, so that $\mathcal{M}_{fgh}$ is a "three variable family" of Galois representations over a triple product of weight spaces

$$
\mathcal{X} = \text{Spec}(\Lambda) \times \text{Spec}(\Lambda) \times \text{Spec}(\Lambda).
$$

A point $(k, \ell, m) \in \mathcal{X}$ is said to be balanced if the three weights are
integers $\geq 1$ whose associated specialisations are classical eigenforms,
and neither weight is greater or equal to the sum of the other two.
The set $\mathcal{X}_{\text{geom}}$ of balanced points is dense in $\mathcal{X}$ for both the rigid
analytic and $p$-adic topologies. For each $x := (k, \ell, m) \in \mathcal{X}_{\text{geom}}$, one
can construct a suitable generalised diagonal cycle in a product of
three Kuga-Sato varieties for weight $k$, $\ell$ and $m$, whose image under
the $p$-adic étale Abel-Jacobi map yields classes

$$
\kappa_x \in H^1(\mathbb{Q}, M_{k\ell m}), \quad M_{k\ell m} := M_{f_k} \otimes M_{g_\ell} \otimes M_{h_m}(t),
$$

where $t$ is chosen so that $M_{k\ell m}$ is the unique Kummer-self-dual Tate
twist of the triple tensor product of the Deligne $p$-adic representations
attached to $f_k$, $g_\ell$, and $h_m$:

$$
M_{k\ell m} = \text{Hom}(M_{k\ell m}, \mathbb{Q}_p(1)).
$$
In order to view $M_{k\ell m}$ as the specialisation at $x$ of $M_{fgh}$, one redefines

$$M_{fgh} := (M_f \otimes M_g \otimes M_h) \otimes \Lambda \Lambda,$$

the last triple product being taken relative to a suitable inclusion $\Lambda \subset \Lambda \otimes \Lambda \otimes \Lambda$. The self-dual restriction on the family $M_{fgh}$ is forced on us by the nature of the construction of the class $\kappa$, and implies that $M_{fgh}$ admits no subfamily of the form $M \otimes \Lambda$, endowed with an “independent” cyclotomic variable.

In addition to the class $\kappa \in H^1(\mathbb{Q}, M_{fgh})$, the setting also comes with three distinct regions of classical interpolation, each giving rise to their own ostensibly different $p$-adic $L$-functions:

- The region $X_f$ of triples $(k, \ell, m) \in (\mathbb{Z}^2)^3$, where $k \geq \ell + m$, where the $p$-adic $L$-function $L^f(M_{fgh})$ interpolates the ratios

$$L^f(M_{k\ell m}) := \frac{\langle f_k, \delta^t g h_m \rangle}{\langle f_k, f_k \rangle}, \quad k = 2t + \ell + m,$$

- The region $X_g$ of triples $(k, \ell, m)$ where $\ell \geq k + m$, where $L^g(M_{fgh})$ interpolate the ratios

$$L^g(M_{k\ell m}) := \frac{\langle g_\ell, \delta^t f_k h_m \rangle}{\langle g_\ell, g_\ell \rangle}, \quad \ell = 2t + k + m,$$

- The region $X_h$ of triples $(k, \ell, m)$ where $m \geq k + \ell$, where $L^h(M_{fgh})$ interpolate the ratios

$$L^h(M_{k\ell m}) := \frac{\langle h_m, \delta^t f_k g_\ell \rangle}{\langle h_m, h_m \rangle}, \quad m = 2t + k + \ell.$$

The class $\kappa$ “ties together” these three $p$-adic $L$-functions, following the “explicit reciprocity laws” of [DR1]. In that sense it could be envisaged as the “mother of all $p$-adic $L$-functions” in the triple product setting.

A particularly simple and appealing case from the point of view of the Birch and Swinnerton-Dyer conjecture arises when the triple $(f, g, h)$ of Hida families specialises in weights $(2, 1, 1) \in X_f$ to a classical triple $(f, g, h) := (f_2, g_1, h_1)$, where $f$ is the weight two newform attached to an elliptic curve $E$, and $g$ and $h$ are classical, hence associated to odd two-dimensional Artin representations $V_g$ and $V_h$. The
Kummer-self-duality assumption implies that $g$ and $h$ have inverse nebentype characters, so that $V_{gh} := V_g \otimes V_h$ has real traces and determinant one. Let $H/\mathbb{Q}$ be the finite Galois extension through which the Artin representation $V_{gh}$ factors. In that setting, the global class $\kappa_x$ can be parlayed into the proof of the following theorem:

**Theorem 3.1 (Rotger, D)** If $L(E, V_{gh}, 1) \neq 0$, then $(E(H) \otimes V_{gh})^{G_\mathbb{Q}}$ is trivial.

**Sketch of proof.** Letting $x := (2,1,1)$, which lies in $\mathcal{X}_f$ and not in $\mathcal{X}_{\text{geom}}$, the explicit reciprocity law of [DR2] relates the $L$-value $L^f(M_x, 0) \sim L(E, V_{gh}, 1)$ to the image of $\kappa_x$ under the dual exponential map, which describes the image of its restriction to $\mathbb{Q}_p$ in the quotient $H^1(\mathbb{Q}_p, V_{fgh})/H^1_f(\mathbb{Q}_p, V_{fgh})$. The non-vanishing of $L(E, V_{gh}, 1)$ implies that the class $\kappa_x$ is “singular” at $p$, and a general argument (involving local and global Tate duality, and the Poitou-Tate long exact sequence) implies the triviality of $(E(H) \otimes V_{gh})^{G_\mathbb{Q}}$ which maps to $H^1_f(\mathbb{Q}, V_{fgh})$ via the connecting homomorphism of Kummer theory.

4 The generalised Perrin-Riou conjecture

Let $p$ be a prime for which the automorphic representations attached to $f$, $g$ and $h$ are unramified. Let $\alpha_g$ and $\beta_g$ denote the roots of the Hecke polynomial $x^2 - a_p(\pi_g)x + \chi_{\pi_g}(p)$ attached to $\pi_g$, ordered in such a way that $\alpha_g$ is the $U_p$-eigenvalue attached to $g$. Similar definitions are made for the invariants $\alpha_h$ and $\beta_h$ associated to $h$. The Artin representations attached to $g$ and $h$ decompose (uniquely, if $\alpha_g \neq \beta_g$ and $\alpha_h \neq \beta_h$) into a direct sum of $G_{\mathbb{Q}_p}$-eigenspaces

$$V_g = V_g^\alpha \oplus V_g^\beta, \quad V_h = V_h^\alpha \oplus V_h^\beta, \quad V_{gh} = V_{gh}^{\alpha\alpha} \oplus V_{gh}^{\alpha\beta} \oplus V_{gh}^{\beta\alpha} \oplus V_{gh}^{\beta\beta}.$$ 

Let $v_{\alpha\alpha}, \ldots, v_{\beta\beta}$ be an eigenbasis for $V_{gh}$.

The “generalised Perrin-Riou conjecture” (as formulated and extended in an ongoing work with Alan Lauder [DL]) takes up the scenario of the previous section, assuming this time that $L(E, V_{gh}, 1) = 0$. We can associate to the pair $(g, h)$ of Hida families a $p$-adic family of overconvergent modular forms of weight two, defined for $k \geq 2$ by

$$\zeta_k(g, h) := d^{1-k} g_k^{[p]} \times h_k,$$
where \( g_k \) and \( h_k \) are the weight \( k \) specialisations of \( g \) and \( h \) respectively, \( g_k^{[p]} \) denotes the so-called \( p \)-depletion of \( g_k \) whose Fourier coefficients are supported on the integers prime to \( p \), and \( d = q \frac{d}{dq} \) is the Atkin-Serre differential operator on \( p \)-adic modular forms. The operator \( d^{1-k} \) maps forms of weight \( k \) to forms of weight \( 2 - k \) and preserves overconvergence, hence \( \zeta_k(g, h) \) is an overconvergent form of weight two, for all \( k \geq 1 \). This overconvergent modular form can be viewed as a rigid differential on a wide open neighbourhood of the \( p \)-ordinary locus of \( X_0(N) \), where \( N \) is prime to \( p \) but chosen to be large enough so that it is divisible by the levels of \( f, g \) and \( h \). (The running self-duality assumption implies that \( \zeta_k(g, h) \) has trivial nebentypus character.) Each \( \zeta_k(g, h) \) for \( k \geq 1 \) represents, via a standard description of Coleman, to a class in the de Rham cohomology of \( X_0(N) \). The work of Andreatta and Iovita implies that the classes \( \zeta_k(g, h) \) enjoy good \( p \)-adic variational properties, as might be naively expected by examining their Fourier coefficients which belong to \( \Lambda \), and interpolates to:

\[
\zeta(g, h) \in H^1_{dR}(X_0(N), \mathbb{C}_p) \otimes \tilde{\Lambda}
\]

where \( \tilde{\Lambda} \supset \Lambda \) consists of power series that converge in the \( p \)-adic open unit disc.

**Definition 4.1** The family \( \zeta(g, h) \) is the Perrin-Riou \( p \)-adic \( L \)-function attached to the pair \((g, h)\) of Hida families. Its projection to the \( f \)-isotypic component \( D_f \otimes \tilde{\Lambda} \), where \( D_f := D_{dR}(M_f) \), is called the Perrin-Riou \( p \)-adic \( L \)-function attached to the triple \((f, g, h)\).

This definition is guided by Perrin-Riou’s philosophy that \( p \)-adic \( L \)-functions attached to \( f \) ought to be viewed as taking values in the Dieudonné module of \( f \). Perrin-Riou’s original study focussed on the case where \( g \) and \( h \) are Hida families of Eisenstein series. In that case the Perrin-Riou \( L \)-function is a refinement of the Mazur-Swinnerton-Dyer \( p \)-adic \( L \)-function attached to the family of cyclotomic twists of \( M_f \); the fact that it takes values in the Dieudonné module of \( M_f \) rather than being scalar valued makes various constructions seem more natural.

Note that when \( k = 1 \), the rigid differential \( \zeta_1(g, h) \) is just the product \( g^{[p]}h \) of classical weight one forms, and hence is the restriction to the ordinary locus of a classical form of level \( Np^2 \). This can be used
to show that
\[
\frac{(1 - \varphi^{-2})}{\mathcal{E}(g, h, \varphi)} \zeta_1(f, g, h) \text{ belongs to } \text{Fil}^1(D_f),
\]
where \(\varphi\) is the Frobenius endomorphism acting on \(D_f\),
\[
\mathcal{E}(g, h, x) := (1 - \alpha_g \alpha_h x) \times (1 - \alpha_g \beta_h x) \times (1 - \beta_g \alpha_h x) \times (1 - \beta_g \beta_h x),
\]
and \(\text{Fil}^1\) denotes the middle step in its Hodge filtration. The Garret-Rankin-Harris-Kudla formula for triple product \(L\)-functions can be used to show that
\[
\frac{(1 - \varphi^{-2})}{\mathcal{E}(g, h, \varphi)} \zeta_1(f, g, h) \sim \left( \frac{L(f, V_{gh}, 1)^{1/2}}{\langle f, f \rangle} \right) \times \omega_f,
\]
where \(\langle \cdot, \cdot \rangle\) is the Petersson scalar product, \(\omega_f\) is the class in \(\text{Fil}^1(D_f)\) attached to \(f\), and \(\sim\) indicates an equality up to a non-zero factor in \(L^\times\).

**Remark:** The fact that \(\zeta_1(f, g, h)\) is also related, via the Kato-Perrin-Riou explicit reciprocity law, to the dual exponential of a “generalised Kato class” \(\kappa(f, g, h)\) is what leads to the proof of the BSD result of the previous section.

If \(L(E, V_{gh}, 1) = 0\), the class \(\zeta_1(g, h)\) vanishes and it is natural to consider the first derivative with respect to \(k\),
\[
\zeta'_1(f, g, h) \in D_f.
\]
Note that the overconvergent form \(\zeta_k(g, h)\) is no longer classical for \(k \geq 1\), and there is no reason to expect \(\zeta'_1(g, h)\) to lie in a simple Frobenius translate of the middle step in the Hodge filtration. The generalised Perrin-Riou conjecture concerns the image of such a translate in \(D_f/\text{Fil}^1D_f\). Recall that Poincaré duality identifies this quotient with the dual of \(\text{Fil}^1(D_f)\).

**Conjecture 4.2** The class \(\zeta'_1(f, g, h)\) satisfies
\[
\left\langle \frac{(1 - \varphi^{-2})}{\mathcal{E}(g, h, \varphi)} \zeta'_1(f, g, h), \omega_f \right\rangle \sim \text{Reg}_{\text{PR}}(E, V_{gh}),
\]
where \(\sim\) denotes an equality up to \(L^\times\) and \(\text{Reg}_{\text{PR}}(E, V_{gh})\) denotes the Perrin-Riou regulator attached to the pair \((E, V_{gh})\) that will be defined in the next section.
5 \( p \)-adic Regulators

Standard “genericity” assumptions on \((f_2, g_1, h_1)\) —for instance, that the three conductors of the associated newforms are relatively prime—imply that the sign in the functional equation governing the parity of the order of vanishing of \(L(E, V_{gh}, s)\) at the central point \(s = 1\) is always equal to 1, and hence that this order of vanishing is even. Assume that

\[
\text{ord}_{s=1} L(E, V_{gh}, s) = 2.
\]

An equivariant version of the Birch and Swinnerton-Dyer conjecture implies that \((E(H) \otimes V_{gh})^{\mathbb{Q}}\) is two-dimensional over the field \(L\) of coefficients of the Artin representation \(V_{gh}\). Let

\[
\begin{align*}
P &= P_{\alpha\alpha}v_{\beta\beta} + P_{\alpha\beta}v_{\beta\alpha} + P_{\beta\alpha}v_{\alpha\beta} + P_{\beta\beta}v_{\alpha\alpha}, \\
Q &= Q_{\alpha\alpha}v_{\beta\beta} + Q_{\alpha\beta}v_{\beta\alpha} + Q_{\beta\alpha}v_{\alpha\beta} + Q_{\beta\beta}v_{\alpha\alpha}
\end{align*}
\]

be a basis for this two dimensional vector space. The element \(P_\xi\) and \(Q_\xi\) are points in \(E(H) \otimes L\) on which the Frobenius element at \(p\) acts with the eigenvalue \(\xi\). The basis \((P, Q)\) is only well-defined up to \(\text{GL}_2(L)\). It may be desirable to refine the choice so that it is well-defined up to \(\text{GL}_2(O)\) where \(O\) is a subring of \(S\)-integral elements, for a well-controlled set of primes \(S\) (presumably, dividing only the order of \(\text{Gal}(H/\mathbb{Q})\)). But we have not attempted such an integral refinement.

The basis \((P, Q)\) gives rise to various \(p\)-adic regulators, referred to as the “Perrin-Riou regulator” and the “DLR regulator”.

The Perrin-Riou regulator is defined to be

\[
\text{Reg}_{\text{PR}}(E, V_{gh}) = \det \begin{pmatrix}
\log_{E,p}(P_{\alpha\alpha}) & \log_{E,p}(P_{\beta\beta}) \\
\log_{E,p}(Q_{\alpha\alpha}) & \log_{E,p}(Q_{\beta\beta})
\end{pmatrix},
\]

(1)

where \(\log_{E,p} : E(\bar{\mathbb{Q}}_p) \to \bar{\mathbb{Q}}_p\) is the formal group logarithm on \(E/\mathbb{Q}_p\). The pair \((\alpha_g, \alpha_h)\) of Frobenius eigenvalues for \(g\) and \(h\) that is privileged in this definition arises from the ordering of \((\alpha, \beta)\) pairs that was fixed at the outset, based on the \(U_p\)-eigenvalues for \(g\) and \(h\).

**Remark:** For the sake of completeness\(^1\), it is worth noting that there is another regulator, the so-called DLR-regulator, which occurs in the

\(^1\)this remark is not used in the discussion that follows, and would only be of interest for a niche audience
elliptic Stark conjectures of [DLR], and privileges the choice of a stabilisation $g_\alpha$ of one of the forms $g$ or $h$. It is defined by the formula

$$\text{Reg}_{g_\alpha}(E, V_{gh}) = \det \begin{pmatrix} \log_{E,p}(P_{\alpha\alpha}) & \log_{E,p}(P_{\alpha\beta}) \\ \log_{E,p}(Q_{\alpha\alpha}) & \log_{E,p}(Q_{\alpha\beta}) \end{pmatrix}. \quad (2)$$

To give a feeling for what these regulators can look like, we will focus on some naturally arising special cases where the representation $V_{gh}$ is reducible:

1. The CM case where $g$ and $h$ are theta series attached to characters $\psi_g$ and $\psi_h$ of the same imaginary quadratic field $K$, with inverse central characters. This implies that the characters $\psi_1 := \psi_g \psi_h$ and $\psi_2 := \psi_g' \psi_h$, where the superscript $'$ denotes the composition with the involution of $\text{Gal}(K/\mathbb{Q})$, are ring class characters, namely, $\psi_1' = \psi_1^{-1}$ and $\psi_2' = \psi_2^{-1}$. The representation $V_{gh}$ breaks up as a direct sum of two odd induced representations:

$$V_{gh} = V_1 \oplus V_2 := \text{Ind}_K^\mathbb{Q}(\psi_1) \oplus \text{Ind}_K^\mathbb{Q}(\psi_2). \quad (3)$$

2. The RM case where $g$ and $h$ are theta series attached to characters $\psi_g$ and $\psi_h$ of mixed signature of the same real quadratic field $K$, with inverse central characters. Just as before, the representation $V_{gh}$ breaks up as a direct sum of two induced representations attached to ring class characters $\psi_1$ and $\psi_2$, which can be ordered uniquely so that $\psi_1$ is totally even and $\psi_2$ is totally odd:

$$V_{gh} = V_1 \oplus V_2 := \text{Ind}_K^\mathbb{Q}(\psi_1) \oplus \text{Ind}_K^\mathbb{Q}(\psi_2).$$

3. Most germane to [HV] is the adjoint case, where $h = g^*$ is the dual of $g$. In that case,

$$V_{gh} = V_1 \oplus V_2,$$

where $V_1$ is the trivial one-dimensional representation and $V_2$ is the trace zero adjoint of $V_g$.

In all these cases, the two-dimensional $L$-vector space $(E(H) \otimes V_{gh})^{G_\mathbb{Q}}$ decomposes as a direct sum of two (possibly trivial) subspaces

$$(E(H) \otimes V_{gh})^{G_\mathbb{Q}} = (E(H) \otimes V_1)^{G_\mathbb{Q}} \oplus (E(H) \otimes V_2)^{G_\mathbb{Q}}.$$
We say that we are in a *rank* \((1, 1)\) *setting* if this decomposition is non trivial, i.e., if both of the vector spaces on the right are one-dimensional. In that case the basis \((P, Q)\) used to define the various regulators can be chosen to be compatible with this decomposition. The following table summarises the shape that the regulator takes in the rank \((1, 1)\) scenario.

<table>
<thead>
<tr>
<th>Setting</th>
<th>(\text{Reg}<em>{\text{PR}}(E, V</em>{gh}))</th>
<th>(\text{Reg}<em>{\text{g}</em>{\alpha}}(E, V_{gh}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>RM/CM, (p) split</td>
<td>0</td>
<td>(\log(P_{\alpha}) \log(Q_{\alpha\beta}))</td>
</tr>
<tr>
<td>RM/CM, (p) inert</td>
<td>(\log(P^+) \log(Q^+))</td>
<td>(\log(P^+) \log(Q^-) - \log(P^-) \log(Q^+))</td>
</tr>
<tr>
<td>Adjoint</td>
<td>(\log(P) \log(Q_1))</td>
<td>(\log(P) \log(Q_{\alpha/\beta}))</td>
</tr>
</tbody>
</table>

The conjectures associated to the entries in the first row are well understood in the CM case. Namely, the derivative \(\zeta'_1(g, h)\) vanishes in the rank \((1, 1)\) CM case, because the family \(\zeta_k(g, h)\) vanishes identically, consistent with the vanishing of the \(p\)-adic regulator. The elliptic Stark conjecture of [DLR] in the CM setting where \(p\) splits in \(K\) corresponds to the second entry on the first row, and was proved in [DLR] by exploiting Heegner points. The RM case presents tantalising new difficulties: in the DLR setting when \(p\) is split in \(K\), they arise from the circumstance that the weight one point on the eigencurve attached to \(g_{\alpha}\) is not étale over weight space, as observed by Cho-Vatsal and Bellaiche-Dimitrov.

The entries in the second row and first column, related to the case where the prime \(p\) is inert in the associated quadratic field, is very poorly understood. The article [DR3] tackles it in the RM scenario where \(f\) has (split or non-split) multiplicative reduction at \(p\), which forces the presence of a “double exceptional zero” of the associated \(p\)-adic \(L\)-function. The second derivative of \(\zeta_k(g, h)\) can then be related to a product of logarithms of idoneous *Stark-Heegner points*. (A similar strategy can certainly be carried out in the CM case, replacing Stark-Heegner points by Heegner points. The CM setting has the advantage that more can be proved thanks to the theory of Heegner points, with the concomittant drawback that the setting presents less mystery and hence the theorems are perhaps less interesting. In any case the CM result is probably not in th literature at present.)

The conjectures involving the second row and second column have been tested extensively (these were the first cases of the “elliptic Stark conjecture” that the authors of [DLR] cut their experimental teeth on) but that case is very poorly understood on the theoretical level, both
in the CM and RM cases, and the complicated nature of the DLR regulator, which fails to factor as a product of logarithms, reflects this complexity.

The conjectures involving the third row are, of course, the most germane to [HV]. In the elliptic setting where $f$ is cuspidal, they seem rather deep, since no reasonable approach is available to independently produce the global point $Q$ whose existence is predicted by the BSD conjecture, except in the case where $g$ is a CM form, which places us back in a setting already covered in the previous rows.

On the other hand one has greater hope of being able to say something non-trivial in the "Eisenstein adjoint setting", when the cusp for $f$ is replaced by an Eisenstein series of weight two. This corresponds to the exact $p$-adic counterpart of the setting considered in [HV].

6 Tame analogues

We keep all the notations that were in force in the previous sections.

Let $q$ be an odd prime that does not divide $N$, let $p$ be another prime that divides $\#E(\mathbb{F}_{q^2})$, and suppose that there is a surjective "discrete elliptic logarithm"

$$\log^q_{E,p}: E(\mathbb{F}_{q^2}) \to \mathbb{Z}/p\mathbb{Z}.$$ 

Since $E(\mathbb{F}_{q^2})$ has cardinality $(q + 1 - a_q(E)) \times (q + 1 + a_q(E))$, the existence of this discrete elliptic logarithm implies that $q$ is a level-raising prime mod $p$, i.e., that there is a weight two form $f_q$ of level $Nq$ which is new at $q$ and congruence to $f$ modulo $p$. The abelian variety attached to this form has split multiplicative reduction at $q$ if $p$ divides $q + 1 - a_q$, and has non-split multiplicative reduction if $p$ divides $q + 1 + a_q$. We will assume that $p$ does not divide both, i.e., that it is odd and relatively prime to $q + 1$.

The discrete Perrin-Riou regulator $\text{Reg}^q_{\text{PR}}(E, V_{gh})$ and the discrete DLR regulators $\text{Reg}^q_{\phi^q}(E, V_{gh})$ are defined by the same formulas as in (1) and (2) respectively, but replacing the formal group logarithm $\log_{E,p}$ by the discrete elliptic logarithm $\log^q_{E,p}$, and the $\varphi$-eigenvalues $(\alpha_q, \beta_q)$ and $(\alpha_h, \beta_h)$ by the eigenvalues for the frobenius element at $q$ (acting on the same Artin representations, but viewed as having mod $p$ coefficients). For the Perrin-Riou regulator, one then wishes to
assume that either 
\[ \alpha_g \alpha_h = \beta_g \beta_h = 1 \quad \text{or} \quad \alpha_g \alpha_h = \beta_g \beta_h = 1. \]

Note that the discrete regulators depend \textit{quadratically} on the choice of discrete logarithm, i.e., replacing \( \log_{E,p}^q \) by the multiple \( t \cdot \log_{E,p}^q \) with \( t \in \mathbb{F}_p^\times \) has the effect of multiplying the regulators by \( t^2 \).

Let \( SS(N; q) \) denote the set of supersingular points on \( X_0(N)/\mathbb{F}_q \), and let \( \mathcal{F}(N; q) \) denote the set of \( \mathbb{Z}/p\mathbb{Z} \)-valued functions on \( SS(N; q) \). Composing the modular parametrisation

\[ \varphi_E : X_0(N)(\mathbb{F}_{q^2}) \to E(\mathbb{F}_{q^2}) \]

with the discrete logarithm \( \log_{E,p}^q \) gives an element \( \phi_{f_q} \in \mathcal{F}(N; q) \) satisfying

\[ T_\ell(\phi_{f_q}) = a_\ell(f_q) \phi_{f_q} \pmod{p}, \quad \text{for all} \ \ell \nmid Npq. \]

Since the spectrum of the Hecke algebra acting on \( \mathcal{F}(N; q) \) captures exactly the systems of eigenvalues attached to modular forms on \( \Gamma_0(Nq) \) which are new at \( q \), the class \( \phi_f \) provides a geometric way of realising the level raising from \( f \) to \( f_q \). We can now state the “tamely ramified Perrin-Riou conjecture”

**Conjecture 6.1 (Tamely ramified Perrin-Riou conjecture)** For all level raising primes \( q \),

\[ \frac{\langle \phi_{f_q}, \phi_{f_q} \rangle \times \langle f_q, g(z)h(qz) \rangle}{\langle f_q, f_q \rangle} \sim \text{Reg}^q_{\text{PR}}(E, V_{gh}), \]

where \( \sim \) denotes equality up to a factor in the localisation of \( \mathcal{O}_L \) at \( p \) which does not depend on \( q \).

Note that both left and right hand sides depend quadratically on the choice of \( \log_{E,p}^q \), and therefore that the truth of this assertion does not depend on this choice.

The main evidence we have for this conjecture at present is the following theorem:

**Theorem 6.2** The tamely ramified Perrin-Riou conjecture above is true in the CM setting where \( g \) and \( h \) are theta series attached to characters of the same imaginary quadratic field.
Note that the running self-duality assumption implies that \( g \) and \( h \) have the same nebentypus character, and hence that \( V_{gh} \) decomposes as in (3) into a direct sum of representations induced from ring class characters. The global points that arise in the tamely ramified conjecture are constructed from (what else?) Heegner points defined over the associated ring class fields.

The proof of this tamely ramified Perrin-Riou conjecture in the CM setting grew out of exchanges with Harris, Rotger and Venkatesh, and adapt equally well to prove the conjecture that was tested numerically in [HV]. The latter corresponds to the situation where \( f \) is replaced by an Eisenstein series of weight two. The proof in this case is actually somewhat more delicate, owing to the more subtle nature of the periods involved in the Eisenstein setting. Instead of an eigenvector in \( F(N;q) \), in the Eisenstein setting it becomes necessary to construct a \textit{generalised eigenvector} which is annihilated by the square of Mazur’s Eisenstein ideal.

7 Conclusion

The possibility of formulating a tamely ramified, Mazur-Tate variant of the Perrin-Riou conjecture is an encouraging sign that the collection of diagonal cycles might possess Euler System like properties, even if these properties fall somewhat outside the more “traditional” Euler system framework.

References


