A Term-Structure Model for Dividends and Interest Rates

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Joint work with Damir Filipović

swiss : finance : institute

School and Workshop on Dynamical Models in Finance
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Overview

1 Introduction

2 Dividend Futures and Bonds

3 Dividend Paying Stock

4 Empirical Analysis

5 Derivative Pricing
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A new market for dividend derivatives

- How can we trade dividends?
  - Synthetic replication.
  - Dividend swaps (OTC) or dividend futures (on exchange).
  - Latest innovations: single names, options, dividend-rates hybrids, . . .

- Asset pricing: term structure of equity risk premium.

- Dividend derivative pricing.

- Derivative pricing with dividend paying stock

- Interest rates: hybrid products, long maturity dividend claims.
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State Process

- Filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})\), with \(\mathbb{Q}\) the risk-neutral pricing measure.
- Multivariate state process \(X_t\) in \(E \subseteq \mathbb{R}^d\) with linear drift:

\[
dX_t = \kappa(\theta - X_t)dt + dM_t,
\]

for \(\kappa \in \mathbb{R}^{d \times d}\), \(\theta \in \mathbb{R}^d\), and some \(d\)-dimensional martingale \(M_t\).
- First moment is linear in the state:

\[
\mathbb{E}_t \left[ \begin{pmatrix} 1 \\ X_T \end{pmatrix} \right] = e^{A(T-t)} \begin{pmatrix} 1 \\ X_t \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 \\ \kappa \theta & -\kappa \end{pmatrix}, \quad t \leq T.
\]
Dividend Futures

- Instantaneous dividend rate:

\[ D_t = p + q^\top X_t, \]

for \( p \in \mathbb{R}, q \in \mathbb{R}^d \) such that \( p + q^\top x \geq 0 \) for all \( x \in E \).

- Dividend futures price:

\[ D_{\text{fut}}(t, T_1, T_2) = \mathbb{E}_t \left[ \int_{T_1}^{T_2} D_s \, ds \right] = (p \quad q^\top) \int_{T_1}^{T_2} e^{A(s-t)} \, ds \left( \begin{array}{c} 1 \\ X_t \end{array} \right). \]

- If \( \kappa \) is invertible:

\[ D_{\text{fut}}(t, T_1, T_2) = (T_2 - T_1) \left( p + q^\top \theta \right) - q^\top \kappa^{-1} \left( e^{-\kappa(T_2-t)} - e^{-\kappa(T_1-t)} \right) (X_t - \theta). \]
Figure: Monthly dividend payments by Eurostoxx 50 constituents (in index points) from October 2009 until October 2016. Source: Eurostoxx 50 DVP index, Bloomberg.
Dividend Seasonality

- Standard choice to model annual cycles:
  \[ \delta(t) = \rho_0 + \rho^\top \Gamma(t), \quad \Gamma(t) = \begin{pmatrix} \sin(2\pi t) \\ \cos(2\pi t) \\ \vdots \\ \sin(2\pi Kt) \\ \cos(2\pi Kt) \end{pmatrix}. \]

- Remark, \( \Gamma(t) \) is the solution of a linear ODE:
  \[ d\Gamma(t) = \text{blkdiag} \left( \begin{pmatrix} 0 & 2\pi \\ -2\pi & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 2\pi K \\ -2\pi K & 0 \end{pmatrix} \right) \Gamma(t) dt. \]

  \( \rightarrow \) We can add \( \Gamma \) to the state vector!

- For example:
  \[ dX_t = \kappa(\delta(t) - X_t) dt + dM_t \]
Interest Rates

- Risk-neutral discount factor:
  \[ \zeta_t = \zeta_0 e^{-\int_0^t r_s \, ds}, \quad t \geq 0, \]
  where \( r_t \) denotes the short rate.

- Directly specify dynamics for \( \zeta_t \):
  \[ \zeta_t := e^{-\gamma t} Y_t, \quad dY_t = \lambda (\phi + \psi^\top X_t - Y_t) dt, \]
  for \( \phi, \lambda, \gamma \in \mathbb{R} \) and \( \psi \in \mathbb{R}^d \) such that \( Y_t > 0 \).

Bond Prices

- Time-\(t\) price of zero-coupon bond maturing at \(T\):

\[
P(t, T) = \frac{1}{\zeta_t} \mathbb{E}_t[\zeta_T] = \frac{e^{-\gamma(T-t)}}{Y_t} \text{e}_d^\top \text{e}^{B(T-t)} \begin{pmatrix} 1 \\ X_t \\ Y_t \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ \kappa \theta & -\kappa & 0 \\ \lambda \phi & \lambda \psi^\top & -\lambda \end{pmatrix}.
\]

- Implied short rate:

\[
r_t = \gamma + \lambda - \lambda \frac{\phi + \psi^\top X_t}{Y_t}.
\]

- If all eigenvalues of \(\kappa\) have positive real parts and \(\lambda > 0\):

\[
\lim_{T \to \infty} -\frac{\log(P(t, T))}{T - t} = \gamma.
\]
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GARCH Diffusion

- Specify martingale part $M_t$ as follows

$$dX_t = \kappa(\theta - X_t)\,dt + \text{diag}(X_{1t}, \ldots, X_{dt})\Sigma \,dB_t,$$

(1)

with $B_t$ a standard $d$-dimensional Brownian motion and $\Sigma \in \mathbb{R}^{d \times d}$ lower triangular with $\Sigma_{ii} > 0$.

- Used before for stochastic volatility (Nelson (1990), Barone-Adesi et al. (2005)), energy markets (Pilipović (1997)), interest rates (Brennan and Schwartz (1979)), and Asian option pricing (Linetsky (2004)).

- Attractive features:
  - Unique positive solution.
  - Flexible correlation structure.
  - Moments in closed-form.
Proposition

Consider the following system of SDEs:

\[
\begin{cases}
\text{d}X_t = \kappa(\theta - X_t)\text{d}t + \text{diag}(X_{1t}, \ldots, X_{dt})\Sigma \text{d}B_t, \\
\text{d}Y_t = \lambda(\phi + \psi^\top X_t - Y_t)\text{d}t
\end{cases}
\]

If \((X_0, Y_0) \in \mathbb{R}^{d+1}_+\), and

\[(\kappa \theta)_i, \psi_i \geq 0, \quad \lambda, \phi \geq 0, \quad \kappa_{ij} \leq 0 \text{ for } i \neq j,\]

then the system has a unique strong solution in \(\mathbb{R}^{d+1}_+\).
Moment Formula

- \((X_t, Y_t)\) is a polynomial diffusions, cfr. Filipović and Larsson (2015).
- Fix basis for \(\text{Pol}_n(\mathbb{R}^d \times \mathbb{R})\), \(n \geq 1\):
  \[
  H_n(x, y) = (1, h_1(x, y), \ldots, h_{N_n}(x, y))^\top,
  \]
  with \(N_n = \binom{n+d+1}{n} - 1\).
- There exists matrix \(G_n\) such that for any \(z \in \text{Pol}_n(\mathbb{R}^d \times \mathbb{R})\)
  \[
  \mathbb{E}_t [z(X_T, Y_T)] = \bar{z}^\top e^{G_n(T-t)} H_n(X_t, Y_t),
  \]
  where \(\bar{z}\) is the coordinate representation of \(z\) with respect to the chosen basis.
- Efficient algorithms exist for \(\bar{z}^\top e^{G_n(T-t)}\), e.g. Al-Mohy and Higham (2011).
Absence of arbitrage:

\[
S_t = \frac{1}{\zeta_t} \mathbb{E}_t[\zeta_T S_T] + \frac{1}{\zeta_t} \mathbb{E}_t \left[ \int_t^T \zeta_s D_s \, ds \right], \quad \forall T \geq t.
\]

If \( \Re(eig(G_2)) < \gamma \), then

\[
\frac{1}{\zeta_t} \mathbb{E}_t \left[ \int_t^\infty \zeta_s D_s \, ds \right] = \frac{1}{Y_t} \tilde{v}^\top (\gamma I - G_2)^{-1} H_2(X_t, Y_t) < \infty,
\]

with \( \tilde{v} \) the coordinate vector of \( (x, y) \mapsto y(p + q^\top x) \) wrt \( H_2(x, y) \).

Stock price representation:

\[
S_t = \frac{L_t}{\zeta_t} + \frac{1}{\zeta_t} \mathbb{E}_t \left[ \int_t^\infty \zeta_s D_s \, ds \right], \quad t \geq 0,
\]

for some non-negative martingale \( L_t \), cfr. Buehler (2010).
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Data

- Eurostoxx 50 stock index
- Eurostoxx 50 dividend futures
  - Underlying: sum of declared ordinary gross cash dividends (or cash equivalent) with ex-date during one calendar year, divided by index divisor on ex-date.
  - Fixed maturity dates in Dec of each year, up to 10y.
  - Example: Today we can trade in maturities Dec(17+k), k = 0, ..., 9. Payoff of Dec(17+k) contract is sum of dividends in [Dec(17+(k−1)), Dec(17+k)].
  - We use 2nd, 3rd, 4th, 5th, 7th, and 10th contract.
- Euribor interest rate swaps
  - Fixed leg pays annual, floating leg semi-annual.
  - Fixed time to maturities: 1,2,3,5,7, and 10 years.
- Daily observations from October 2009 until October 2016, \((1 + 6 + 6) \times 1827 = 23,751\) observations in total.
### Summary Statistics

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Median</th>
<th>Std</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Swap rates (%)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 yrs</td>
<td>0.66</td>
<td>0.42</td>
<td>0.63</td>
<td>−0.23</td>
<td>2.03</td>
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<tr>
<td>2 yrs</td>
<td>0.76</td>
<td>0.52</td>
<td>0.72</td>
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<tr>
<td>3 yrs</td>
<td>0.91</td>
<td>0.67</td>
<td>0.80</td>
<td>−0.25</td>
<td>2.79</td>
</tr>
<tr>
<td>5 yrs</td>
<td>1.25</td>
<td>1.07</td>
<td>0.91</td>
<td>−0.18</td>
<td>3.22</td>
</tr>
<tr>
<td>7 yrs</td>
<td>1.57</td>
<td>1.44</td>
<td>0.96</td>
<td>−0.03</td>
<td>3.50</td>
</tr>
<tr>
<td>10 yrs</td>
<td>1.94</td>
<td>1.87</td>
<td>0.97</td>
<td>0.24</td>
<td>3.77</td>
</tr>
<tr>
<td><strong>Dividend futures</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1-2 yrs</td>
<td>109.42</td>
<td>110.30</td>
<td>6.63</td>
<td>87.10</td>
<td>125.30</td>
</tr>
<tr>
<td>2-3 yrs</td>
<td>104.93</td>
<td>106.80</td>
<td>8.88</td>
<td>81.20</td>
<td>119.90</td>
</tr>
<tr>
<td>3-4 yrs</td>
<td>101.57</td>
<td>103.10</td>
<td>9.95</td>
<td>74.10</td>
<td>120.60</td>
</tr>
<tr>
<td>4-5 yrs</td>
<td>99.60</td>
<td>100.80</td>
<td>10.50</td>
<td>70.80</td>
<td>122.90</td>
</tr>
<tr>
<td>6-7 yrs</td>
<td>96.85</td>
<td>97.40</td>
<td>11.85</td>
<td>69.90</td>
<td>126.50</td>
</tr>
<tr>
<td>9-10 yrs</td>
<td>95.69</td>
<td>95.80</td>
<td>14.19</td>
<td>63.00</td>
<td>132.50</td>
</tr>
<tr>
<td><strong>Eurostoxx 50 index</strong></td>
<td>2,877.62</td>
<td>2,890.35</td>
<td>363.15</td>
<td>1,995.01</td>
<td>3,828.78</td>
</tr>
</tbody>
</table>
Model Specification

- We model \( X_t = (X_t^I, X_t^D) \) and set \( d = 2 \times 2: \)

\[
\begin{align*}
\mathrm{d}X_{1t}^I &= \kappa_1^I (X_{2t}^I - X_{1t}^I) \, \mathrm{d}t + X_{1t}^I \Sigma_{11}^I \, \mathrm{d}B_{t}^\mathbb{P} \\
\mathrm{d}X_{2t}^I &= \kappa_2^I (\theta^I - X_{2t}^I) \, \mathrm{d}t + X_{2t}^I \Sigma_{22}^I \, \mathrm{d}B_{t}^\mathbb{P} \\
\mathrm{d}X_{1t}^D &= \kappa_1^D (X_{2t}^D - X_{1t}^D) \, \mathrm{d}t + X_{1t}^D \Sigma_{11}^D \, \mathrm{d}B_{t}^\mathbb{P} \\
\mathrm{d}X_{2t}^D &= \kappa_2^D (\theta^D - X_{2t}^D) \, \mathrm{d}t + X_{2t}^D \Sigma_{22}^D \, \mathrm{d}B_{t}^\mathbb{P}
\end{align*}
\]

with \( B_{t}^\mathbb{P} \) a 4-dimensional \( \mathbb{P} \)-Brownian motion and

\[
\Sigma = \begin{pmatrix}
\Sigma_{11}^I \\
\Sigma_{12}^I \\
\Sigma_{11}^D \\
\Sigma_{22}^D
\end{pmatrix} = \begin{pmatrix}
\Sigma_{11}^I & \Sigma_{12}^I & 0 & 0 \\
0 & \Sigma_{22}^I & 0 & \Sigma_{12}^D \\
\Sigma_{11}^D & 0 & \Sigma_{22}^D & 0 \\
0 & 0 & 0 & \Sigma_{22}^D
\end{pmatrix}, \quad \psi = \begin{pmatrix}1 \\ 0 \\ 0 \\ 0\end{pmatrix}, \quad q = \begin{pmatrix}0 \\ 0 \\ 1 \\ 0\end{pmatrix}.
\]

- Constant market price of risk vector \( \Lambda \in \mathbb{R}^4: \)

\[
\mathbb{E}_t^\mathbb{P} \left[ \frac{\mathrm{d}Q}{\mathrm{d}P} \right] = \exp \left( \Lambda^\top B_t^\mathbb{P} - \frac{1}{2} \| \Lambda \|^2 t \right).
\]

- Restrictions on parameters such that: \( (X_t, Y_t) > 0, \)

\[
\lim_{T \to \infty} E^\mathbb{P}[H_2(X_T, Y_T)] < \infty, \text{ and } S_t < \infty, \quad \forall t > 0.
\]
Parameter Estimation

- Quasi-maximum likelihood + Kalman filtering.
- Transition equation: \( Z_t = \Phi_0 + \Phi_1 Z_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, q(Z_{t-1})) \).
- Measurement equation: \( M_t = h(Z_t) + \nu_t, \quad \nu_t \sim \mathcal{N}(0, \sigma_M^2 \text{Id}) \).
  - Dividend futures: linear.
  - Swap rates and stock price: non-linear.
    \( \rightarrow \) Unscented Kalman filter.

- All observations are scaled by their sample mean.
- Three estimation steps:
  - Estimate \( \kappa_1^D, \kappa_2^D, \theta_D, \Sigma_{11}^D, \Sigma_{22}^D \) from dividend futures.
  - Estimate \( \kappa_1^I, \kappa_2^I, \theta_I, \Sigma_{11}^I, \Sigma_{22}^I \) from swap rates.
  - Re-estimate all parameters using all instruments.
### Parameter Estimates

<table>
<thead>
<tr>
<th></th>
<th>$\varepsilon_1^I$</th>
<th>$\varepsilon_2^I$</th>
<th>$\varepsilon_1^D$</th>
<th>$\varepsilon_2^D$</th>
<th>$\Sigma_{11}$</th>
<th>$\Sigma_{22}$</th>
<th>$\Sigma_{11}^D$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.124*</td>
<td>0.005***</td>
<td>0.122***</td>
<td>0.000</td>
<td>0.017**</td>
<td>0.110***</td>
<td>0.120***</td>
</tr>
<tr>
<td></td>
<td>(0.076)</td>
<td>(0.001)</td>
<td>(0.015)</td>
<td>(0.002)</td>
<td>(0.009)</td>
<td>(0.004)</td>
<td>(0.012)</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th></th>
<th>$\Sigma_{22}$</th>
<th>$\Lambda_1$</th>
<th>$\Lambda_2$</th>
<th>$\Lambda_3$</th>
<th>$\Lambda_4$</th>
<th>$\Sigma_{1,1}^D$</th>
<th>$\Sigma_{2,2}^D$</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>0.162***</td>
<td>-0.413***</td>
<td>-0.300***</td>
<td>-0.426***</td>
<td>0.202***</td>
<td>-0.024**</td>
<td>-0.022**</td>
</tr>
<tr>
<td></td>
<td>(0.011)</td>
<td>(0.040)</td>
<td>(0.030)</td>
<td>(0.082)</td>
<td>(0.042)</td>
<td>(0.011)</td>
<td>(0.010)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$\theta^D$</th>
<th>$\gamma$</th>
<th>$\lambda$</th>
<th>$\sigma_M$</th>
<th>$\mathcal{L} \times 10^{-4}$</th>
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<tbody>
<tr>
<td></td>
<td>0.778***</td>
<td>0.032***</td>
<td>0.132**</td>
<td>0.044***</td>
<td>3.852</td>
</tr>
<tr>
<td></td>
<td>(0.049)</td>
<td>(0.002)</td>
<td>(0.079)</td>
<td>(0.000)</td>
<td></td>
</tr>
</tbody>
</table>

### Table: Quasi-maximum likelihood estimates with asymptotic standard deviations in parenthesis.

### Corresponding instantaneous correlation matrix:

\[
\begin{pmatrix}
X_1^I & X_2^I & X_1^D & X_2^D \\
X_1^I & 1.00 & & \\
X_2^I & 0.00 & 1.00 & \\
X_1^D & -0.19 & 0.00 & 1.00 \\
X_2^D & 0.00 & -0.13 & 0.00 & 1.00
\end{pmatrix}
\]
## Error Analysis

<table>
<thead>
<tr>
<th>Maturities</th>
<th>Swap rates</th>
<th>Dividend futures</th>
<th>Eurostoxx 50</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>All</td>
<td>1 y</td>
<td>2 y</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>RMSE (bps)</td>
<td>6.62</td>
<td>3.98</td>
<td>4.59</td>
</tr>
<tr>
<td>MAE (bps)</td>
<td>4.85</td>
<td>3.24</td>
<td>3.55</td>
</tr>
<tr>
<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

**Table:** The first five days of the sample are dropped when computing the error statistics to give the Kalman filter time to learn the current value of $X_t$. The remaining sample period consists of 1,822 daily observations between October 8, 2009 and October 1, 2016.
Filtered State

(a) Interest rate factors

(b) Dividend factors
Filtered Swap Rates

Maturity 1 yrs

Maturity 2 yrs

Maturity 3 yrs

Maturity 5 yrs

Maturity 7 yrs

Maturity 10 yrs
Filtered Dividend Futures Prices

- Maturity 1-2 yrs
- Maturity 2-3 yrs
- Maturity 3-4 yrs
- Maturity 4-5 yrs
- Maturity 6-7 yrs
- Maturity 9-10 yrs

Sander Willems (SFI@EPFL)  A TSM for Dividends and Interest Rates  May 24th 2017
<table>
<thead>
<tr>
<th>Dividend spot</th>
<th>Mean (%)</th>
<th>Std (%)</th>
<th>Sharpe</th>
<th>β</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 yrs</td>
<td>0.23</td>
<td>3.02</td>
<td>0.08</td>
<td>0.71</td>
</tr>
<tr>
<td>3 yrs</td>
<td>0.17</td>
<td>2.82</td>
<td>0.06</td>
<td>0.85</td>
</tr>
<tr>
<td>4 yrs</td>
<td>0.12</td>
<td>2.73</td>
<td>0.04</td>
<td>0.96</td>
</tr>
<tr>
<td>5 yrs</td>
<td>0.07</td>
<td>2.71</td>
<td>0.03</td>
<td>1.06</td>
</tr>
<tr>
<td>6 yrs</td>
<td>0.04</td>
<td>2.73</td>
<td>0.01</td>
<td>1.13</td>
</tr>
<tr>
<td>7 yrs</td>
<td>0.02</td>
<td>2.77</td>
<td>0.01</td>
<td>1.19</td>
</tr>
<tr>
<td>10 yrs</td>
<td>-0.02</td>
<td>2.86</td>
<td>-0.01</td>
<td>1.30</td>
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</table>

<table>
<thead>
<tr>
<th>Bonds</th>
<th>Mean (%)</th>
<th>Std (%)</th>
<th>Sharpe</th>
<th>β</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 yrs</td>
<td>0.02</td>
<td>0.13</td>
<td>0.15</td>
<td>0.02</td>
</tr>
<tr>
<td>3 yrs</td>
<td>0.03</td>
<td>0.22</td>
<td>0.14</td>
<td>0.04</td>
</tr>
<tr>
<td>4 yrs</td>
<td>0.04</td>
<td>0.32</td>
<td>0.13</td>
<td>0.06</td>
</tr>
<tr>
<td>5 yrs</td>
<td>0.06</td>
<td>0.44</td>
<td>0.13</td>
<td>0.09</td>
</tr>
<tr>
<td>6 yrs</td>
<td>0.07</td>
<td>0.56</td>
<td>0.12</td>
<td>0.12</td>
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<tr>
<td>7 yrs</td>
<td>0.08</td>
<td>0.68</td>
<td>0.12</td>
<td>0.15</td>
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<tr>
<td>10 yrs</td>
<td>0.11</td>
<td>1.03</td>
<td>0.11</td>
<td>0.23</td>
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</table>

<table>
<thead>
<tr>
<th>Index</th>
<th>Mean (%)</th>
<th>Std (%)</th>
<th>Sharpe</th>
<th>β</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.05</td>
<td>2.09</td>
<td>0.02</td>
<td>1.00</td>
</tr>
</tbody>
</table>

**Table:** Monthly returns in excess of the 1-month risk-free rate.
Overview

1. Introduction
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Derivative Pricing with Polynomial Expansions

- Denote $Z_t = (X_t, Y_t)$ and $Z = (Z_{t_1}, \ldots, Z_{t_n})$ for some finite time partition $t_1 < \cdots < t_n$.
- Time-$t$ price of a (path dependent) derivative:
  \[
  \pi_t = \mathbb{E}_t[F(Z)],
  \]
  for some discounted payoff function $F$ on $E = (\mathbb{R}^d \times \mathbb{R})^n$.
- Denote by $g(dz)$ the (unknown) conditional distribution of $Z$.
- Let $w(dz)$ be an auxiliary distribution such that $g(dz) \ll w(dz)$ and
  \[
  g(dz) = \ell(z)w(dz).
  \]
- Define Hilbert space $L^2_w(E)$ with norm and scalar product
  \[
  \|f\|^2_w = \int_E f(z)^2 w(dz), \quad \langle f, h \rangle_w = \int_E f(z)h(z)w(dz).
  \]
Derivative Pricing with Polynomial Expansions

Assumptions:

1. \( \text{Pol}(E) \subset L^2_w \),
2. \( \ell \in L^2_w \),
3. \( F \in L^2_w \),
4. \( g \ll w \).

Let \( H = \{ H_0(z) = 1, H_1(z), H_2(z), \ldots \} \) be an orthonormal set of polynomials spanning the closure \( \text{Pol}(E) \) in \( L^2_w \).

Let \( \bar{F} \) be the orthogonal projection of \( F \) onto \( \overline{\text{Pol}(E)} \) in \( L^2_w \).

Elementary functional analysis now gives:

\[
\bar{\pi}_t = \mathbb{E}[\bar{F}(Z)] = \langle \bar{F}, \ell \rangle_w = \sum_{k \geq 0} F_k \ell_k,
\]

\[
F_k = \langle \bar{F}, H_k \rangle_w = \langle F, H_k \rangle_w = \int_E F(z) H_k(z) w(dz),
\]

\[
\ell_k = \langle \ell, H_k \rangle_H = \mathbb{E}_t[H_k(Z)].
\]
Truncating the series for $\overline{\pi}$, we get:

$$\pi_t^{(K)} = \sum_{k=0}^{K} F_k l_k$$

$$= \pi_t + \left( \overline{\pi}_t - \pi_t \right) + \left( \pi_t^{(K)} - \pi_t \right).$$

- $(\overline{\pi}_t - \pi_t) = 0$ if $\text{Pol}(E) = L_w^2$.
- $(\pi_t^{(K)} - \pi_t) \to 0$ as $K \to \infty$.
- Crucial question: how to choose the auxiliary distribution?
The Auxiliary Distribution

- We choose the multivariate log-normal distribution:

**Definition**

A random vector \((X_1, \ldots, X_k) \in \mathbb{R}_+^k, k \geq 1\), is said to have a multivariate log-normal distribution \(\mathcal{LN}(\mu, \Lambda)\) if

\[
\log(X_1), \ldots, \log(X_k) \sim \mathcal{N}(\mu, \Lambda),
\]

for some \(\mu \in \mathbb{R}^k\) and some positive semi-definite \(\Lambda \in \mathbb{R}^{k \times k}\).

- Very easy to simulate from a log-normal.
- Finite moments of any order:

\[
\mathbb{E}[X_1^{\alpha_1} \cdots X_k^{\alpha_k}] = \exp \left\{ \alpha^\top \mu + \frac{1}{2} \alpha^\top \Lambda \alpha \right\} < \infty, \quad \forall \alpha \in \mathbb{N}^k.
\]

\(\rightarrow\) Assumption 1 is always satisfied
- Moment indeterminate \(\rightarrow\) projection bias.
The Auxiliary Distribution

**Theorem**

Let \( m = (d + 1)n \). Suppose that the random vector \( \mathbf{Z} = (Z_{t_1}, \ldots, Z_{t_n}) \) admits a continuous density with support on \( \mathbb{R}^m_+ \). Let \( w \) be the \( \mathcal{LN}(\mu, \Lambda) \) distribution with \( \mu \in \mathbb{R}^m \) and pos. def. \( \Lambda \in \mathbb{R}^{m \times m} \). Define the matrix \( M \in \mathbb{R}^{n \times n} \) as

\[
M = \frac{1}{\bar{\sigma}^2} T^{-1} - (I_n \otimes 1_{d+1}^T) \Lambda^{-1} (I_n \otimes 1_{d+1}),
\]

where \( \bar{\sigma}^2 = \max \{ (\Sigma \Sigma^T)_{ij} \mid i, j = 1, \ldots, d \} \), and \( T = (t_i \wedge t_j)_{1 \leq i, j \leq n} \). If \( M \) is positive semi-definite, then assumption 2 is satisfied.

**Lemma**

Suppose that \( \Sigma \) has strictly positive diagonal elements, \( \lambda \psi \) is different from the zero vector, and \( (X_0, Y_0) \in \mathbb{R}^{d+1}_+ \). Then for any \( t > 0 \), the random vector \( Z_t = (X_t, Y_t) \) has an infinitely differentiable density.
Examples of (discounted) derivative payoffs

- **Swaption:**
  \[
  \frac{\zeta_{T_0}}{\zeta_t} \left( \pi_{T_0}^{\text{swap}} \right)^+ = \frac{e^{-\gamma T_0}}{Y_t} \left( w_{\text{swap}} H_1(X_{T_0}, Y_{T_0}) \right)^+.
  \]

- **Dividend option:** Denote \( l_t = \int_0^t D_s \, ds \),
  \[
  \frac{\zeta_{T_1}}{\zeta_t} \left( \int_{T_0}^{T_1} D_s \, ds - K \right)^+ = \frac{e^{-\gamma (T_1 - t)}}{Y_t} Y_{T_1} \left( l_{T_1} - l_{T_0} - K \right)^+.
  \]

- **Stock option:**
  \[
  \frac{\zeta_T}{\zeta_t} (S_T - K)^+ = \frac{e^{-\gamma (T - t)}}{Y_t} \left( \bar{\nu}^\top \left( \gamma I - G_2 \right)^{-1} H_2(X_T, Y_T) - Y_T K \right)^+.
  \]

- **Dividend-Rates hybrid:**
  \[
  \frac{\zeta_T}{\zeta_t} \left( \int_{T-1}^T D_s \, ds \right)^+ \left( \frac{L_{1y}^T}{S_T} - \bar{L}_{1y}^T \right)^+.
  \]
Thank you for your attention!
References


References II


<table>
<thead>
<tr>
<th>Time to maturity</th>
<th>Open interest</th>
<th>Daily volume</th>
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<tr>
<td></td>
<td>Mean</td>
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**Table:** Open Interest and Daily Volume of Dividend Futures