Approximating the Permanent

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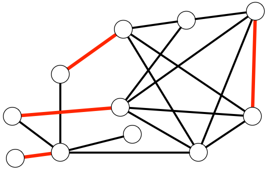
Matchings

Undirected graph $G = (V, E)$:



Matching = subset of vertex disjoint edges.
Perfect Matching = matching of size $n/2$.

Let Ω = collection of all matchings of G (of all sizes).
Let $\mathcal{P}$ = perfect matchings of G.

In time polynomial in $n = |V|$,

- Can we compute $|\Omega| = \#$ of matchings?
  Generate a matching uniformly at random (u.a.r.) from Ω?
- Can we compute $|\mathcal{P}| = \#$ of perfect matchings?
  Generate a perfect matching u.a.r. from $\mathcal{P}$?

$(0,1)$-Permanent = #BIP-Perfect-Match

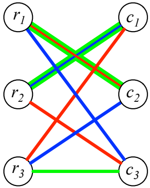
For $n \times n$ non-negative matrix A, permanent:

$$\mathrm{per}(A) = \sum_{\pi} \prod_{i} A(i, \pi(i))$$

where π ranges over all permutations of $\{1, \ldots, n\}$.

For $0 - 1$ matrix, view A as adjacency matrix for bipartite graph.

Example: $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$



$$per(A) = A(1,2)A(2,3)A(3,1) + A(1,3)A(2,1)A(3,2) + A(1,2)A(2,1)A(3,3) = 3$$

Complexity of Computing Permanent

Given a graph $G = (V, E)$ with $n = |V|$ vertices,

let $\mathcal{P}$ = perfect matchings of G.
let Ω = all matchings of G (any size).

Can we compute $|\Omega|$ and $|\mathcal{P}|$ in time polynomial in n?

- Polynomial time algorithm for planar graphs [Kasteleyn '67]
- #P-Complete for bipartite graphs [Valiant '79]
- FPRAS for $|\Omega|$ on general graphs [Jerrum-Sinclair '89]
- FPRAS for $|\mathcal{P}|$ on bipartite graphs [JSV '04]
  Fastest algorithm: $O^*(n^7)$ time
  [Bezakova,Stefankovic,Vazirani,V '09]

For non-negative matrices A:
running time depends on $\log(a_{\max}/a_{\min})$
Strongly polynomial time (indpt. of entries of A):
using [Linial,Samorodnitsky,Wigderson '00]'s matrix scaling

Markov Chain for Matchings

For an undirected graph $G = (V, E)$,
how to sample from $\Omega =$ all matchings.

From $X_t \in \Omega$ the transition $X_t \to X_{t+1}$ is defined as follows:

1. Choose an edge $e = (v, w)$ uniformly at random from E.
2. *Remove:* If $e \in X_t$ then set $X_{t+1} = X_t \setminus \{e\}$.
3. *Add:* If v and w are unmatched in X_t then $X_{t+1} = X_t \bigcup \{e\}$.
4. *Superfluous Slide:*
   If v is unmatched and w is matched (or vice-versa):
   1. Let (w, z) denote the matched edge.
   2. Set $X_{t+1} = X_t \bigcup (v, w) \setminus (w, z)$.
5. Otherwise, set $X_{t+1} = X_t$.

Symmetric and ergodic, unique stationary dist. $\pi = \textit{uniform}(\Omega)$.

Mixing time: $$\mathrm{T_{mix}} = \max_{X_0 \in \Omega} \min\{t : \mathrm{d_{TV}}(P^t(X_0, \cdot), \pi) \leq 1/4\}.$$

Conductance

Underlying directed graph $H = (\Omega, E_{\mathrm{P}})$ of the Markov chain:

Vertices $=$ states Ω

Edges $= E_P := \{M \to M' : M, M' \in \Omega, \mathrm{P}(M, M') > 0\}$.

For a set $S \subset \Omega$ where $\pi(S) \leq 1/2$ define its conductance by:

$$\begin{aligned} \phi(S) &:= \mathbf{Pr}\left(X_{t+1} \notin S \mid X_t \in S, X_t \sim \pi\right) \\ &= \sum_{M \in S, M' \in \overline{S}} \frac{\pi(M)\mathrm{P}(M, M')}{\pi(S)} \end{aligned}$$

For our chain since $P(M, M') = 1/m$, and π is uniform simplifies:

$$\phi(S) = \frac{1}{m} \frac{\#\{\text{of edges from } S \text{ to } \overline{S}\}}{|S|}$$

$$\text{Let } \phi = \min_{S : \pi(S) \leq 1/2} \phi(S)$$

$$\Omega\left(\frac{1}{\phi}\right) = \mathrm{T}_{\mathrm{mix}} = O\left(\frac{1}{\phi^2} \log(1/\pi_{\min})\right).$$

[Lawler,Sokal '88], [Sinclair,Jerrum '89]

Canonical Paths

For every pair $I, F \in \Omega$ define a path $\gamma_{I,F}$ along edges of H.
For matchings chain, $P(M, M') = 1/m$, and $\pi = uniform(\Omega)$.
For edge $T = M \rightarrow M' \in E_\mathrm{P}$, define its *congestion*:

$$\mathrm{cp}(T) = \{(I, F) : T \in \gamma_{I,F}\}$$

$$\text{Let } \rho = \max_{T \in E_\mathrm{P}} \frac{|\mathrm{cp}(T)|}{|\Omega|}$$

$$\text{Claim:} \quad \phi \geq \frac{1}{2m\rho}$$

Proof: For $S \subset \Omega$ where $|S| \leq |\overline{S}|$, bound $E(S, \overline{S})$:
$|S| \times |\overline{S}|$ (I, F) pairs where $I \in S$ and $F \in \overline{S}$.
Each $T \in E_\mathrm{P}$ has at most $\rho|\Omega|$ paths thru it.
Hence, $\geq \frac{|S||\overline{S}|}{\rho\Omega} \geq \frac{|S|}{2\rho}$ transitions from S to $\overline{S}$. □

Corollary: $\mathrm{T}_\mathrm{mix} = O((m\rho)^2 \log(1/\pi_{\min}))$.

Canonical Paths: General Statement

For every pair $I, F \in \Omega$ define a path $\gamma_{I,F}$ along edges of H.
For transition $T = M \to M' \in E_{\mathrm{P}}$, let

$$\rho(T) := \frac{1}{\pi(M)P(M,M')} \sum_{(I,F):T \in \gamma_{I,F}} \pi(I)\pi(F).$$

Aside: for matchings chain,
$P(M,M') = 1/m$, and $\pi = uniform(\Omega)$ so: $\rho(T) = \frac{|cp(T)|}{m}$.

Let $\rho := \max_{T \in E_{\mathrm{P}}} \rho(T)$.

Theorem [Sinclair '91]: $\mathrm{T}_{\mathrm{mix}} = O(\rho \ell_{\max} \log(1/\pi_{\min}))$,

where $\ell_{\max} = \max_{I,F} |\gamma_{I,F}|$ is the length of the longest path $\gamma_{I,F}$.
see also [Diaconis,Stroock '91], [Jerrum, Sinclair '89]

For matchings chain, we'll see: $\rho = O(m)$ and $\ell_{\max} = O(n)$ so:
$\mathrm{T}_{\mathrm{mix}} = O(n^2 m \log n)$.

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Canonical Paths Definition

Consider a pair of matchings I and F.

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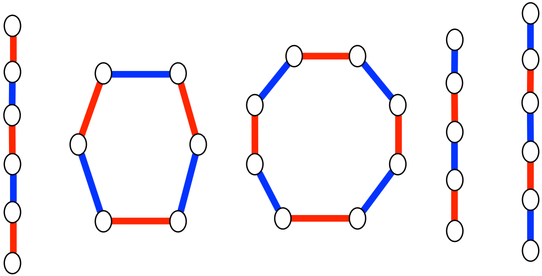
Look at their difference: $I \oplus F$.

Consists of alternating/augmenting paths and alternating cycles:



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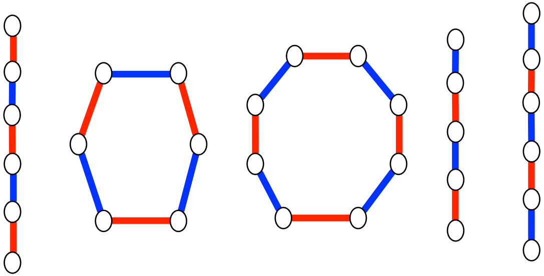

Definition of $\gamma_{I,F}$:

1. Order components of $I \oplus F$ by smallest vertex in each.
2. "Unwind" components in order.

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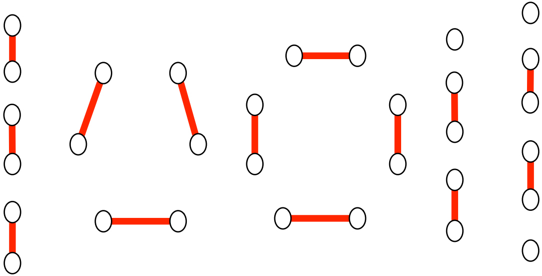

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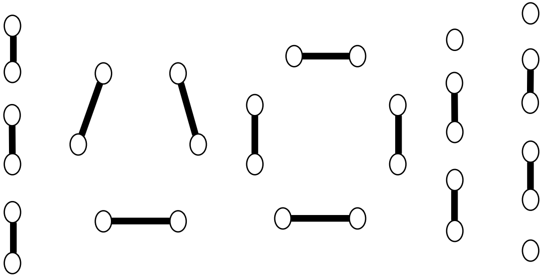

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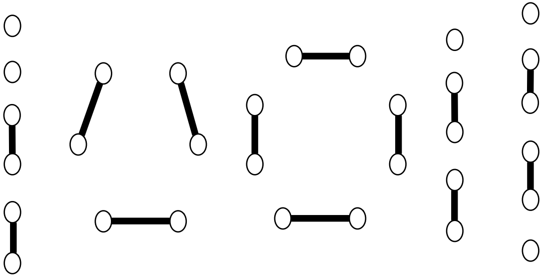

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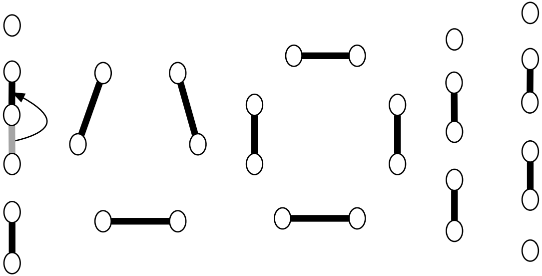

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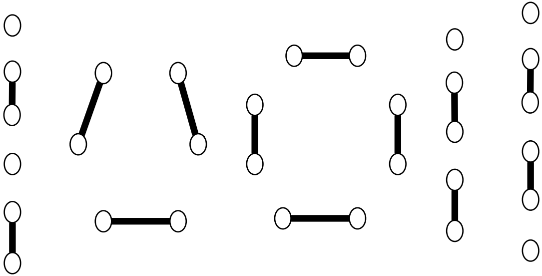

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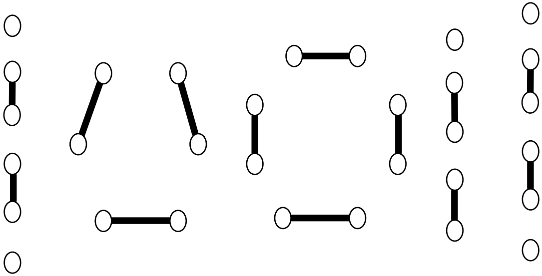

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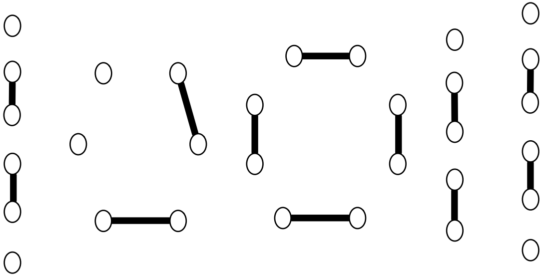

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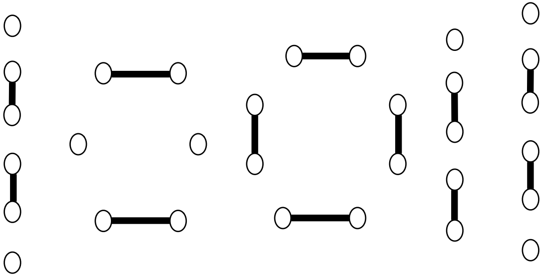

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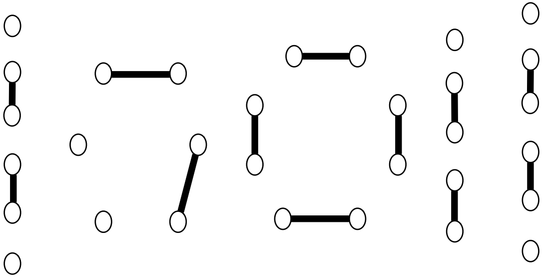

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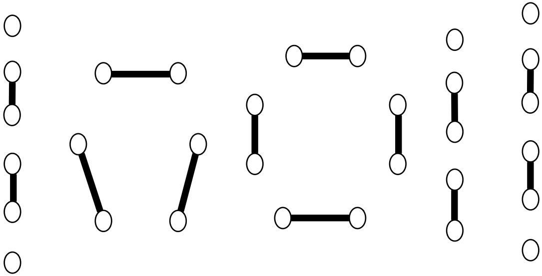

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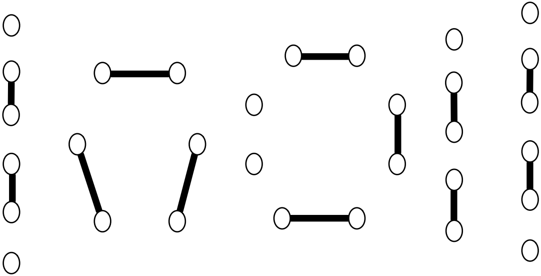

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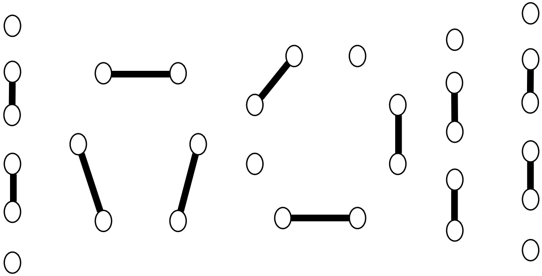

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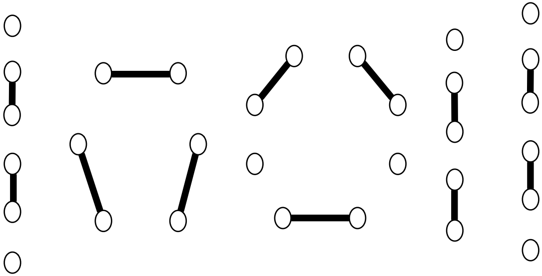

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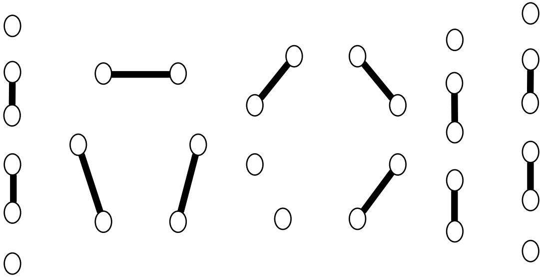

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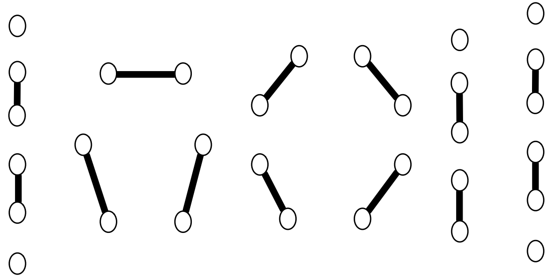

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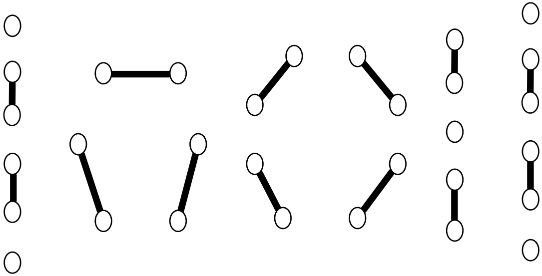

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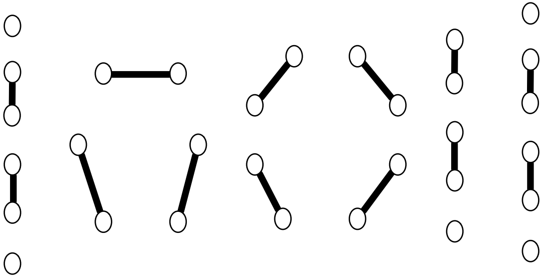

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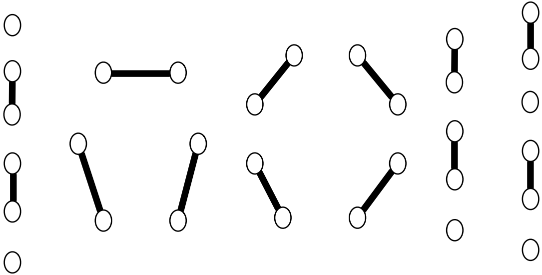

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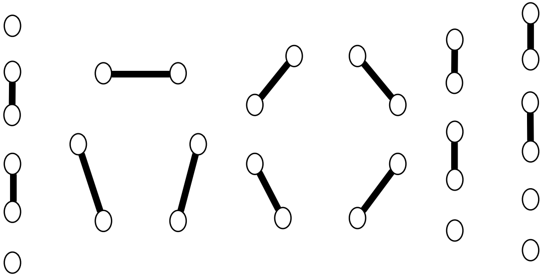

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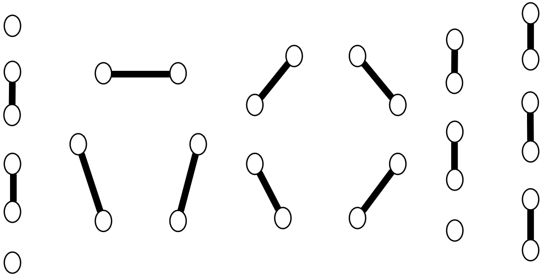

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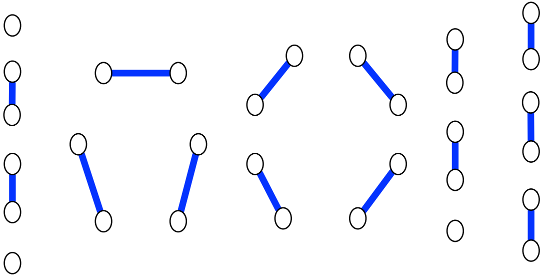

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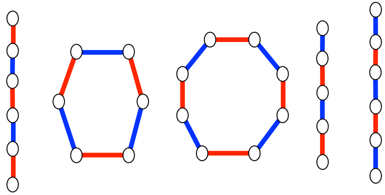
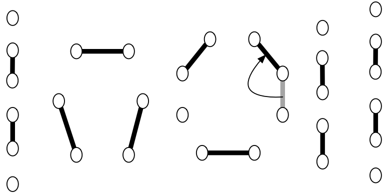
Easy to define η:

$$\eta_T(I, F) = (I \cap F) \bigcup (I \oplus F \setminus (M \cup M'))$$

Example I and F:



Transition $T = M \rightarrow M'$:



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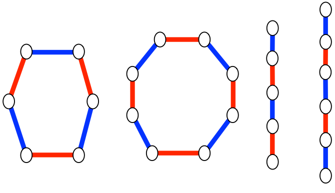

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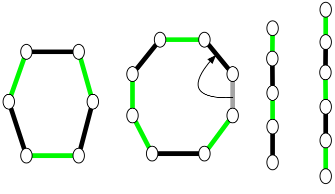

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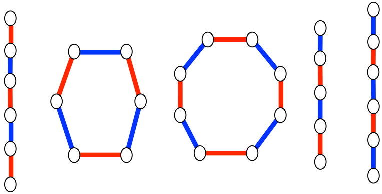

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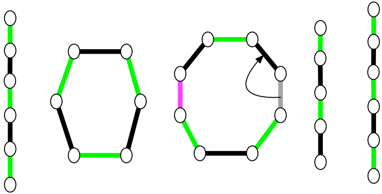

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where e_0 is the first edge of I in the current cycle.

1. Permanent Definition
2. Random Matching
3. Random Perfect Matching

First Idea for Markov Chain

For bipartite graph $G = (V, E)$ with $n + n$ vertices,

$$\text{let } \mathcal{P} = \text{ perfect matchings of } G.$$

Can we design a Markov chain only on $\mathcal{P}$?

What are the transitions?

First Idea for Markov Chain

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Can we design a Markov chain only on $\mathcal{P}$?

What are the transitions?

Enlarge the states: Near-perfect matchings:

$$\text{let } \mathcal{N} = \text{ matchings of } G \text{ with exactly 2 unmatched vertices.}$$

$$\text{Let } \Omega = \mathcal{P} \bigcup \mathcal{N}.$$

Run earlier Markov chain restricted to Ω.

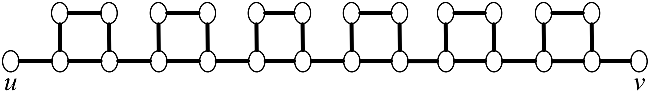
Markov Chain for Perfect Matchings

Consider an undirected bipartite graph $G = (V, E)$.
Let $\Omega = \mathcal{P} \bigcup \mathcal{N}$.

From a matching $X_t \in \Omega$ the transition $X_t \to X_{t+1}$ is defined by:

1. Choose an edge $e = (v, w)$ uniformly at random from E.
2. *Remove:* If $e \in X_t$ and $X_t \in \mathcal{P}$ then set $X_{t+1} = X_t \setminus \{e\}$.
3. *Add:* If v and w are unmatched in X_t then $X_{t+1} = X_t \bigcup \{e\}$.
4. *Slide:* If v is unmatched and w is matched (or vice-versa):
    1. Let (w, z) denote the matched edge.
    2. Set $X_{t+1} = X_t \bigcup (v, w) \setminus (w, z)$.
5. Otherwise, set $X_{t+1} = X_t$.

Bad Example



Key properties:

- $|\mathcal{P}| = 1$: Only 1 perfect matching
- $|\mathcal{N}| \geq 2^{n/4}$: if u and v unmatched then 2^s ways to complete where s is $\#$ of squares.

Conclusion:
Sampling from $\Omega = \mathcal{P} \bigcup \mathcal{N}$ may not help for sampling from $\mathcal{P}$.

What if Perfects are likely?

Suppose $|\mathcal{N}| \leq n^C |\mathcal{P}|$ for constant $C > 0$.
(Example: Graph on n vertices where min-degree $> n/2$.)

Then, MC on $\Omega = \mathcal{P} \bigcup \mathcal{N}$ has $\pi(\mathcal{P}) \geq 1/n^C$.

Can we define canonical paths to prove $\mathrm{T}_{\mathrm{mix}} = O(\mathrm{poly}(n))$?

- $I \in \mathcal{P}$ to $F \in \mathcal{P}$: Same canonical path as before.
- $I \in \mathcal{N}$ to $F \in \mathcal{P}$: $I \oplus F$ is augmenting path + alternating cycles.
  Augment along path to get to a perfect, then unwind cycles.
- $I \in \mathcal{P}$ to $F \in \mathcal{N}$: Unwind cycles then de-augmenting path.
- $I \in \mathcal{N}$ to $F \in \mathcal{N}$: Problem! Might use 4-holes.

Solution: For each $A \in \mathcal{P}$: send $1/|\mathcal{P}|$ units of flow $I \rightarrow A \rightarrow F$.
Congestion increases by $|\mathcal{N}|^2/|\mathcal{P}| \leq n^{2C}$.

Hence, $\mathrm{T}_{\mathrm{mix}} = \mathrm{poly}(n)$.

Weights on Matchings

How to deal with general bipartite graphs?
When $|\mathcal{N}|$ might be huge and $|\mathcal{P}|$ relatively small.

Assign matching $M \in \Omega$ a weight $w(M)$.

Add "Metropolis filter" to the Markov chain so that:

$$\text{Stationary distribution } \pi(M) \propto w(M).$$

Choose weights so that:

1. $\pi(\mathcal{P}) = 1/\text{poly}(n)$ and every $P \in \mathcal{P}$ has the same weight.
2. Markov chain has mixing time $\text{poly}(n)$.

Revised Markov Chain

Consider an undirected bipartite graph $G = (V, E)$.
Let $\Omega = \mathcal{P} \bigcup \mathcal{N}$.

From a matching $X_t \in \Omega$ the transition $X_t \to X_{t+1}$ is defined by:

1. Choose an edge $e = (v, y)$ uniformly at random from E.
2. *Remove:* If $e \in X_t$ and $X_t \in \mathcal{P}$ then set $X' = X_t \setminus \{e\}$.
3. *Add:* If v and y are unmatched in X_t then $X' = X_t \bigcup \{e\}$.
4. *Slide:* If v is unmatched and y is matched (or vice-versa):
   1. Let (y, z) denote the matched edge.
   2. Set $X' = X_t \bigcup (v, y) \setminus (y, z)$.
5. If X' is defined then:
   set $X_{t+1} = X'$ with probability $\min\{1, w(X')/w(X_t)\}$
6. Otherwise, set $X_{t+1} = X_t$.

Stationary distribution: $\pi(M) \propto w(M)$.

Choice of Weights

Weight of matching $M \in \mathcal{P} \cup \mathcal{N}$ depends on unmatched vertices.

If $M \in \mathcal{P}$ then $w(M) = 1$.

Let $\mathcal{N}(u, v) = \{M \in \mathcal{N} : u \text{ and } v \text{ are unmatched}\}$.

If $M \in \mathcal{N}(u, v)$ then $w(M) = w(u, v)$ where:

$$w(u, v) = \frac{|\mathcal{P}|}{|\mathcal{N}(u, v)|}$$

$$\text{Note: } \sum_{P \in \mathcal{P}} w(P) = \sum_{N \in \mathcal{N}(u,v)} w(N) = |\mathcal{P}|$$

Hence: $\pi(\mathcal{P}) = \pi(\mathcal{N}(u, v)) = 1/(n^2 + 1)$.

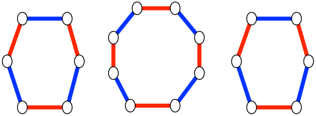
In other words, π is uniform over $n^2 + 1$ hole patterns.

Rapid Mixing

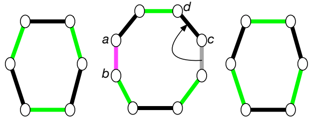
Key: for $I, F \in \Omega$, for $T = M \to M' \in \gamma_{I,F}$,

$$w(I)w(F) \le w(M)w(\eta_T(I,F)).$$

Example $I \in \mathcal{P}$ and $F \in \mathcal{P}$:



Transition $T = M \to M'$:



Note, transition $M \in \mathcal{N}(b,d)$ and Encoding $\eta \in \mathcal{N}(a,c)$.

Need: $w(I)w(F) \le w(M)w(\eta) \iff |\mathcal{P}|^2 \ge |\mathcal{N}(a,c)| \times |\mathcal{N}(b,d)|$

where $(a,b),(c,d) \in E$.

True for bipartite graphs!

Wrong Weights

Summary: For weights $w^*(u,v) = \frac{|\mathcal{P}|}{|\mathcal{N}(u,v)|}$, we have:

- $\mathrm{T}_{\mathrm{mix}} = \mathrm{poly}(n)$.
- $\pi^*(\mathcal{P}) = \pi^*(\mathcal{N}(u,v)) = 1/(n^2+1)$, and

What if we instead use weights $\widehat{w}(u,v)$ where:

$$\frac{1}{2}\frac{|\mathcal{P}|}{|\mathcal{N}(u,v)|} \leq \widehat{w}(u,v) \leq 2\frac{|\mathcal{P}|}{|\mathcal{N}(u,v)|}$$

Then,

- $\mathrm{T}_{\mathrm{mix}}$ slows by $O(1)$ so still rapidly mixing since $\widehat{w}(I)\widehat{w}(F) \leq \widehat{w}(T)\widehat{w}(\eta)$ fails by at most factor 16.
- Note, $\widehat{\pi}(\mathcal{N}(u,v)) \propto \widehat{w}(u,v)|\mathcal{N}(u,v)|$ and $\widehat{\pi}(\mathcal{P}) \propto |\mathcal{P}|$.

$$\frac{\widehat{\pi}(\mathcal{N}(u,v))}{\widehat{\pi}(\mathcal{P})} = \widehat{w}(u,v)\frac{|\mathcal{N}(u,v)|}{|\mathcal{P}|} = \frac{\widehat{w}(u,v)}{w^*(u,v)}$$

Therefore, $$w^*(u,v) = \widehat{w}(u,v)\frac{\widehat{\pi}(\mathcal{P})}{\widehat{\pi}(\mathcal{N}(u,v))}$$

Estimate $\widehat{\pi}(\mathcal{P})/\widehat{\pi}(\mathcal{N}(u,v))$ using $O(n^2 \log n)$ samples from $\widehat{\pi}$.

Summary

Ideal weights: $w^*(u,v) = \frac{|\mathcal{P}|}{|\mathcal{N}(u,v)|}$.

If use slightly wrong weights $\widehat{w}(u,v)$ where:

$$\frac{1}{2}\frac{|\mathcal{P}|}{|\mathcal{N}(u,v)|} \leq \widehat{w}(u,v) \leq 2\frac{|\mathcal{P}|}{|\mathcal{N}(u,v)|}$$

then mixing time slows by a factor of 16.

- Generate $O(n^2 \log n)$ samples from $\widehat{\pi}$.

Let $S(u,v)$ be the set of samples in $\mathcal{N}(u,v)$ and S be those in $\mathcal{P}$.

Then, correct weights:

$$w^*(u,v) = \widehat{w}(u,v)\frac{|S|}{|S(u,v)|}$$

Simulated Annealing Approach

Input bipartite graph $G = (L \cup R, E)$ captured by:
complete bipartite $K_{n,n}$ with edge activities for $y \in L, z \in R$:

$$\lambda(y,z) = \begin{cases} \lambda & \text{if } (y,z) \notin E \\ 1 & \text{if } (y,z) \in E \end{cases}$$

Slowly go from $\lambda = 1$ to $\lambda \approx 0$.

Matching M of $K_{n,n}$ has activity: $\lambda(M) = \prod_{(y,z)\in M} \lambda(y,z)$.

$$\text{Redefine } w^*(u,v) = \frac{\lambda(\mathcal{P})}{\lambda(\mathcal{N}(u,v))}.$$

Define sequence of graphs:
Graph G_i defined by $\lambda = \lambda_i := (1 - 1/2n)^i$.
Note, $\lambda_i(M) \geq \lambda_{i+1}(M) \geq \frac{1}{\sqrt{2}}\lambda_i(M)$, hence

$$\frac{1}{2} w^*_{i+1}(u,v) \leq w^*_i(u,v) \leq 2w^*_{i+1}(u,v).$$

Simulated Annealing Approach

Non-edges have weight λ.

Graph G_i defined by $\lambda = \lambda_i := (1 - 1/2n)^i$.

Weights w_i^* for G_i are reasonable approx to w_{i+1}^* for G_{i+1}:

$$\frac{1}{2} w_{i+1}^*(u,v) \leq w_i^*(u,v) \leq 2 w_{i+1}^*(u,v).$$

Starting graph: $G_0 = K_{n/2,n/2}$

Weights are easy to compute: $w_0^*(u,v) = n$.

Ending graph: G_ℓ for $\ell = 2n^2 \ln n$

Then $\lambda_\ell < 1/n!$ and hence $G_\ell \approx G$.

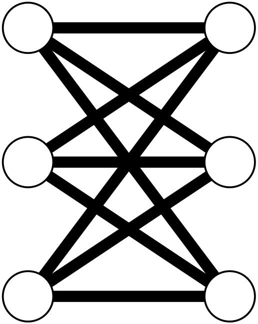
Algorithm:

1. Set $\lambda_0 = 1$. For all $u \in L, v \in R$ set $w_0^*(u,v) = n$.
2. For $i = 0 \to \ell - 1$:
   1. Run MC on G_{i+1} using weights w_i^*.
      Generate $O(n^2 \log n)$ samples from $\pi = \pi_{G_{i+1}, w_i^*}$.
   2. For all $u \in L, v \in R$, set:

$$w_{i+1}^* = w_i^* \frac{\pi(\mathcal{P})}{\pi(\mathcal{N}(u,v))}$$

Simulated Annealing Algorithm

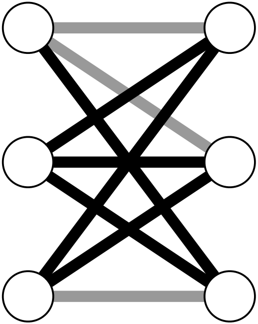
Illustration of the algorithm:



$$weights = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$$

1. Start at the complete bipartite graph
2. Slowly remove non-edges:
    - Generate many samples from π, and
    - Recalibrate the weights $w(u, v)$.

Illustration of the algorithm:



$$weights = \begin{bmatrix} 2.33 & 2.33 & 1.75 \\ 2.8 & 2.8 & 3.5 \\ 2.33 & 2.33 & 3.5 \end{bmatrix}$$

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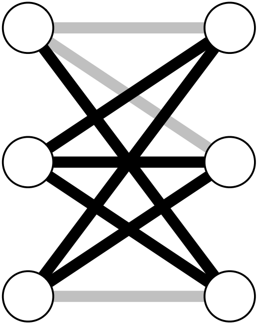
Simulated Annealing Algorithm

Illustration of the algorithm:



$$weights = \begin{bmatrix} 2.1 & 2.1 & 1.31 \\ 2.47 & 2.47 & 5.25 \\ 2.1 & 2.1 & 5.25 \end{bmatrix}$$

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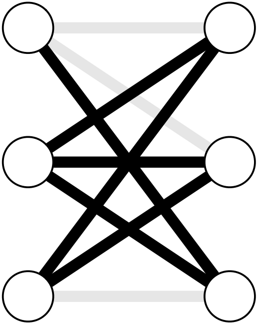
Illustration of the algorithm:



$$\textit{weights} = \begin{bmatrix} 2.03 & 2.03 & 1.14 \\ 2.15 & 2.15 & 9.125 \\ 2.03 & 2.03 & 9.125 \end{bmatrix}$$

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Simulated Annealing Algorithm

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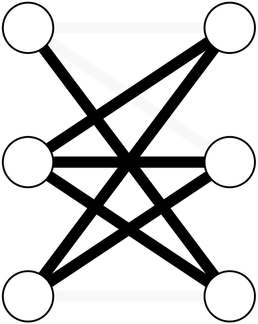

$$weights = \begin{bmatrix} 2.007 & 2.007 & 1.066 \\ 2.124 & 2.124 & 17.06 \\ 2.007 & 2.007 & 17.06 \end{bmatrix}$$

1. Start at the complete bipartite graph
2. Slowly remove non-edges:
   - Generate many samples from π, and
   - Recalibrate the weights $w(u, v)$.

The end

Thank you!