Partition functions and Scaling

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Computing partition functions

"Combinatorics and Complexity of Partition functions", B’2016
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An easy example: $\mathcal{F} = 2^N$. 
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An easy example: $\mathcal{F} = 2^{[N]}$. Here $p_{\mathcal{F}} = \prod_{i=1}^{N} (1 + x_i)$. 
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A more complicated case: the permanent. This will be our main example.
• Permanents.
● Permanents.

● Contingency tables and integer flows.
- Permanents.
- Contingency tables and integer flows.
- Mixed discriminants.
The permanent of an $n \times n$ matrix $A = (a_{ij})$ is

$$\text{Per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i \sigma(i)}$$
Permanent: Definition

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Permanent - examples

\[
\text{Per} \begin{pmatrix}
1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1
\end{pmatrix} = \text{Det} \begin{pmatrix}
1 & 0 & \ldots & 0 \\
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\end{pmatrix} = 1
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Permanent - examples

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\begin{align*}
\text{Per} & \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix} = \text{Det} \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix} = 1 \\
\text{Per} & \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 \end{pmatrix} = n! \neq \text{Det} \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 \end{pmatrix}
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• Examples are not easy to come by...
Permanents and counting

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- In fact, if we can compute permanents, we can compute 'anything'.
Permanents and counting

- If $A$ is the adjacency matrix of a bipartite graph, $\text{per}(A)$ is the number of perfect matchings in the graph.
- Applications in statistical physics if the graph is a grid.
- In fact, if we can compute permanents, we can compute ’anything’.
- So we would like to compute permanents.
How to compute the permanent?

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- Polya: can’t do this by adding signs to the entries. MM’61: Any linear forms, MR’04: not even with quadratic blow-up.
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- What about more general matrices?
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- It would be interesting though to quantify this feeling. **Conjecture AA’10**: Hard to approximate the permanent of a random Gaussian matrix.
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- The best deterministic approximation of $C \approx 2^n$ is obtained in LSW’98, GS’14 via matrix scaling.
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  - And an approach exploiting the similarity of permanent and determinant.
Approximating the permanent via determinants

Old history

- GG’78: To estimate the permanent of $A = (a_{ij})$, let

$$B = \left(\epsilon_{ij} \cdot \sqrt{a_{ij}}\right)$$

$\epsilon_{ij}$ are independent random variables, with mean zero and variance 1.
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- That is, a value \( F(A) \) such that w.h.p.
  \[ F(A) \leq \text{Per}(A) \leq c^n \cdot F(A) \quad c \approx 4 \]
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The important thing about 4 is that it is bigger than \( e \).
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Egorychev, Falikman’81:

$$Per(A) \geq \frac{n!}{n^n} > e^{-n}$$
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- Back to determinants. B’99: take $\epsilon_{ij}$ to be quaternion Gaussian.
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- Back to determinants. **B’99**: take \( \epsilon_{ij} \) to be quaternion Gaussian. A \( 1.2^n \)-approximation.
A simple iterative algorithm to make a matrix doubly-stochastic

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Questions: Does it converge to a doubly stochastic matrix? If yes, how fast? Can we keep track of the permanent?
Sinkorn’s scaling

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Let $r_1, \ldots, r_n$ be the row sums of $A_k$, $k > 1$ even. Then

$$\text{Per}(A_{k+1}) = \frac{1}{\prod_{i=1}^{n} r_i} \cdot \text{Per}(A_k)$$
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- Hence the permanent is trackable. In fact, for every \( i, j, k \):

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- Hence the permanent is trackable. In fact, for every $i, j, k$:

$$A_k(i, j) = \lambda_i^{(k)} A(i, j) \mu_j^{(k)} \Rightarrow \text{Per} (A_k) = \prod_i \lambda_i^{(k)} \cdot \prod_j \mu_j^{(k)} \cdot \text{Per}(A)$$
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Let \( r_1, \ldots, r_n \) be the row sums of \( A_k \), \( k > 1 \) even. Then

\[
\text{Per} (A_{k+1}) = \frac{1}{\prod_{i=1}^{n} r_i} \cdot \text{Per} (A_k) > \text{Per} (A_k)
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• It is also increasing, which indicates that the sequence $\{A_k\}_k$ converges to a doubly stochastic matrix. The rate of convergence can be slow, but this could be handled with some preprocessing.
Matrix scaling

- Let $A = (a_{ij})$ be an $n \times n$ matrix with positive entries. Then
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There exist 'unique' positive numbers \( \lambda_1, \ldots, \lambda_n \) and \( \mu_1, \ldots, \mu_n \) such that the matrix \( (\lambda_i a_{ij} \mu_j) \) is doubly stochastic.
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**Proof:**

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2. Return $\left(\prod_i \lambda_i \cdot \prod_j \mu_j\right)^{-1} \cdot e^{-n}$. Done.
Scaling as an approach

Abstracting out

- Given a partition function $p_F(x_1, ..., x_n) = \sum_{S \in F} \prod_{i \in S} x_i$
Scaling as an approach

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- Given a partition function $p_F(x_1, \ldots, x_n) = \sum_{S \in F} \prod_{i \in S} x_i$:
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  - The function $p_{\mathcal{F}}(y_1, \ldots, y_n)$ is well-behaved (well-concentrated) on its domain. This would usually mean that the transformed object has some regularity properties.
Concentration of the permanent - can we improve it?

- A key fact: if $A$ is doubly stochastic, then $e^{-n} < \text{Per}(A) \leq 1$. 
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- Most matrices are far from both $I$ and $J$. (In fact, rather closer to $J$ than to $I$).
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- Can we "look" at a doubly stochastic matrix and estimate its permanent better?
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- Most matrices are far from both $I$ and $J$.

- Can we "look" at a doubly stochastic matrix and estimate its permanent better? Need "non-blackbox bounds" for the permanent.
A small improvement

- A step in this direction was taken in GS'14. Let $A = (a_{ij})$ be doubly stochastic.
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$$\text{Per}(A) \geq \prod_{i,j=1}^{n} (1 - a_{ij})^{1-a_{ij}}$$
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\text{Per}(A) \leq \prod_{i=1}^{n} \| a_i \|_\psi
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We have

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- The ratio between the bounds is about $2^n$, giving an $2^n$-approximation for the permanent.
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- The upper bound we state is far from being optimal.
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- The ratio between the bounds is about $2^n$, giving an $2^n$-approximation for the permanent.
- The upper bound we state is far from being optimal. We next describe a famous upper bound for $0 - 1$ matrices and some of its applications.
Bregman’s upper bound

• B’73: Let $A$ be a $0 – 1$ matrix with row sums $r_i$
Bregman’s upper bound

- **B’73**: Let $A$ be a $0 – 1$ matrix with row sums $r_i$ then

$$\text{Per}(A) \leq \prod_{i=1}^{n} (r_i!)^{1/r_i}$$
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$$Per(A) \leq \prod_{i=1}^{n} (r_i!)^{1/r_i}$$

- This is tight for a block-diagonal matrix with blocks of size $r_i \times r_i$. 
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• Corollary. If $A$ is a row-stochastic matrix with maximal $i$-th row element $m_i \leq \frac{1}{r_i}$, where $r_i \in \mathbb{N}$, then
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$$\text{Per}(A) \leq \left( \prod_{i=1}^{n} r_i \right)^{-1} \cdot \prod_{i=1}^{n} (r_i!)^{1/r_i}$$
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- **B’73**: Let \( A \) be a \( 0 - 1 \) matrix with row sums \( r_i \) then

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Per(A) \leq \prod_{i=1}^{n} (r_i!)^{1/r_i}
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- **Corollary**. If \( A \) is a row-stochastic matrix with maximal \( i \)-th row element \( m_i \leq \frac{1}{r_i} \), where \( r_i \in \mathbb{N} \), then

\[
Per(A) \leq \left( \prod_{i=1}^{n} r_i \right)^{-1} \cdot \prod_{i=1}^{n} (r_i!)^{1/r_i}
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- This is easy to see by a simple variation argument.
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- Corollary. If $A$ is a row-stochastic matrix with maximal $i$-th row element $m_i \leq \frac{1}{r_i}$, where $r_i \in \mathbb{N}$, then

$$\text{Per}(A) \leq \left( \prod_{i=1}^{n} r_i \right)^{-1} \cdot \prod_{i=1}^{n} (r_i!)^{1/r_i}$$

- Since $r_i! \approx (2\pi r_i)^{1/r_i} \cdot \frac{r_i}{e}$, we see that for large $r_i$ the upper bound we get is close to $e^{-n}$. 
Strong concentration for balanced doubly stochastic matrices

- Let $A$ be a doubly stochastic matrix with maximal entry $t$. 
Strong concentration for balanced doubly stochastic matrices

• Let $A$ be a doubly stochastic matrix with maximal entry $t$. Then

$$e^{-n} < Per(A) \leq \left(\frac{2\pi}{t}\right)^{tn} \cdot e^{-n}$$
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- Let $A$ be a doubly stochastic matrix with maximal entry $t$. Then
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- If $t \leq O(\log(n)/n)$ then $e^{-n} < \text{Per}(A) \leq n^{O(\log(n))} \cdot e^{-n}$.
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- An application: A permanent of a random matrix is easy to approximate within a small factor.
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- I.e., if entries of $A$ are independent standard exponential random variables
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- An application: A permanent of a random matrix is easy to approximate within a small factor.
- I.e., if entries of $A$ are independent standard exponential random variables, then scaling approximates $\text{Per}(A)$ w.h.p. up to a factor of $n^{O(\log(n))}$. 
Proof

- Recall that we compute the scaling factors \( \{\lambda_i\} \) and \( \{\mu_j\} \) of \( A \) and return \( \left( \prod_i \lambda_i \cdot \prod_j \mu_j \right)^{-1} \cdot e^{-n} \).
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- It suffices to show that if \( B = (\lambda_i a_{ij} \mu_j) \) (this is the doubly stochastic part of \( A \)) then \( e^{-n} < Per(B) \leq n^{O(\log(n))} \cdot e^{-n} \).
- The entries of \( A \) have density \( e^{-t}, t \geq 0 \).
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• The entries of \( A \) have density \( e^{-t}, t \geq 0 \). Their expectation is 1 and they are well concentrated.
Proof

• Recall that we compute the scaling factors \( \{\lambda_i\} \) and \( \{\mu_j\} \) of \( A \) and return \( \left( \prod_i \lambda_i \cdot \prod_j \mu_j \right)^{-1} \cdot e^{-n} \).

• It suffices to show that if \( B = (\lambda_i a_{ij} \mu_j) \) (this is the doubly stochastic part of \( A \)) then \( e^{-n} < \text{Per}(B) \leq n^{O(\log(n))} \cdot e^{-n} \).

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- It suffices to show that if \( B = (\lambda_i a_{ij} \mu_j) \) (this is the doubly stochastic part of \( A \)) then \( e^{-n} < Per(B) \leq n^{O(\log(n))} \cdot e^{-n} \).
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- Therefore w.h.p. the scaling factors \( \{\lambda_i\} \) and \( \{\mu_j\} \) of \( A \) are all about \( 1/\sqrt{n} \), and the maximal entry of \( B \) is at most \( \log(n)/n \).
- Done.
- Permanents.
- Contingency tables and integer flows.
- Mixed discriminants.
An application - counting contingency tables and integer flows

- A contingency table is an $n \times n$ integer matrix with prescribed row and column sums.
An application - counting contingency tables and integer flows

- A contingency table is an $n \times n$ integer matrix with prescribed row and column sums. (In this talk consider only square contingency tables.)
A contingency table is an $n \times n$ integer matrix with prescribed row and column sums.

Counting the number $|\Sigma(R, C)|$ of contingency tables with row sums $R$ and column sums $C$ is interesting, because of applications in statistics, combinatorics, and other areas.
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Since exact counting is $\#P$ hard [DKM '94], the realistic goal is to approximate this number.
An application - counting contingency tables and integer flows

- A contingency table is an $n \times n$ integer matrix with prescribed row and column sums.
- We want to approximate the number $|\Sigma(R, C)|$ of contingency tables with row sums $R$ and column sums $C$.
- Connection to partition functions: Given a set of nonnegative weights $W = \{w_{ij}\}_{1 \leq i, j \leq n}$, let

$$p_{\Sigma(R, C)}(W) = p_{\Sigma(R, C)}(w_{11}, \ldots, w_{nn}) = \sum_{D=(d_{ij})} \prod_{i,j=1}^{n} w_{ij}^{d_{ij}}$$
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This is a multiset partition function $B'16$. Note that

$|\Sigma(R, C)| = p_{\Sigma(R, C)}(1, \ldots, 1)$. 
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- This is a multiset partition function $B'16$. Note that $|\Sigma(R, C)| = p_{\Sigma(R,C)}(1, \ldots, 1)$.
- For 0–1 weights $W$, $p_{\Sigma(R,C)}(W)$ counts the number of integer flows in bipartite graphs.
Good’s heuristic for $|\Sigma(R, C)|$

- Let $R = (r_1, \ldots, r_n)$ and $C = (c_1, \ldots, c_n)$. 
Good’s heuristic for $|\Sigma(R, C)|$

- Let $R = (r_1, ..., r_n)$ and $C = (c_1, ..., c_n)$. Let $N = \sum_i r_i = \sum_j c_j$. 
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- Let $R = (r_1, ..., r_n)$ and $C = (c_1, ..., c_n)$. Let $N = \sum_i r_i = \sum_j c_j$.
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- Similarly for columns.
- Good’s heuristic G’76: both events are essentially independent. Hence

$$
|\Sigma(R, C)| \approx \left( \frac{N + n^2 - 1}{n^2 - 1} \right)^{-1} \cdot \prod_{i=1}^{n} \binom{r_i + n - 1}{n - 1} \cdot \prod_{j=1}^{n} \binom{c_j + n - 1}{n - 1}
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• **Theorem**: There is an absolute constant $\gamma > 0$ and an efficiently computable quantity $\rho(R, C, W)$ such that
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Approximating the partition function
Barvinok’07–’16

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• This gives the best known **rigorous** approximation for $|\Sigma(R, C)|$ for many regimes of row and column sums.

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A key ingredient: strong concentration of the permanent for balanced doubly stochastic matrices.
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Barvinok’07–’16

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• In particular, Good’s heuristic is wrong. For typical margins $R$ and $C$, the row and column events are positively correlated.

• A key ingredient: strong concentration of the permanent for balanced doubly stochastic matrices. Will describe this for a very special case.
A very special case - magic squares

- A magic square is a contingency table in which all row and column sums equal a predetermined sum $s$. 
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- Note that $\Sigma(1) = S_n$. Hence $p_{\Sigma(1)}(W) = Per(W)$.
- We describe an $n^{O(\log(n))}$-approximation algorithm for $|\Sigma(s)| = p_{\Sigma(s)}(1, \ldots, 1)$ from BLSY'07.
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- This algorithm, with minor changes, extends to balanced weights $W$. 
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- We describe an $n^{O(\log(n))}$-approximation algorithm for $|\Sigma(s)| = \rho_{\Sigma(s)}(1, \ldots, 1)$ from BLSY’07.
- This algorithm, with minor changes, extends to balanced weights $W$, when all the weights are within a constant factor from each other.
A very brief overview of the algorithm

- B’03: Represent $\Sigma(R, C)$ as an expectation of the permanent of a random matrix with exponentially distributed entries.
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- Show that the bad part is negligible.
A very brief overview of the algorithm

- B’03: Represent $\Sigma(R, C)$ as an expectation of the permanent of a random matrix with exponentially distributed entries.
- Split corresponding integral into two parts - Good, corresponding to balanced matrices, and Bad.
- Show that the bad part is negligible.
- Compute the good part within a factor of $n^{O(\log(n))}$, using strong concentration of the permanent of the doubly stochastic part of the scaled matrix.
Representation $|\Sigma(n)|$ as an integral

- From now on restrict to a case of $n \times n$ magic squares with row and column sums $n$. 
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    b_{11}C & b_{12}C & \ldots & b_{1n}C \\
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- Bang’s identity B’77, F’78:

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\text{Per}(B \otimes J) = (n!)^{2n} \cdot \sum_{\alpha \in \Sigma(n)} \prod_{i,j=1}^{n} b_{i,j}^{\alpha_{ij}} / \alpha_{ij}!
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- Hence B’03 (also in much higher generality)

$$|\Sigma(\bar{n})| = (n!)^{-2n} \cdot \int_{\mathbb{R}^n_+} \text{Per}(B \otimes J)d(B)$$
Representation $|\Sigma(n)|$ as an integral

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  |\Sigma(\bar{n})| = (n!)^{-2n} \cdot \int_{\mathbb{R}^{n^2}_+} \text{Per}(B \otimes J) d(B) = (n!)^{-2n} \cdot \int_{\mathbb{R}^{n^2}_+} \text{Per}(B \otimes J) \exp\{- \sum b_{i,j}\} db_{1,1}...db_{n,n}
  \]
Estimating the integral

- Let $N = n^2$. For an $N \times N$ matrix $A$ let $S(A)$ be the doubly stochastic part of $A$ and $\sigma(A)$ be the inverse product of the scaling factors of $A$. 
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- Write

$$|\Sigma(n)| = \int_{\text{good } B} \text{Per}(B \otimes J)d(B) + \int_{\text{bad } B} \text{Per}(B \otimes J)d(B) =: I_g + I_b$$
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- We will argue that $I_b$ is negligible and that $I_g$ is easy to approximate.
The bad integral is small

- We know $A$ is good if the maximal element of $S(A)$ is of order $O(\log(N)/N)$. 
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- It is somewhat harder to guarantee properties of $S(B \otimes J)$ compared to $S(B)$ since tensor product amplifies bad events.
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- Turns out to suffice to require that the sum of maximal row elements in $S(B \otimes J)$ is of order $O(\log N)$. 
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- Bad integral

$$I_b = \int_{\text{bad} B} \text{Per}(B \otimes J) d(B)$$

is over bad matrices $B$ - with large sum of maximal elements.
The bad integral is small

- Turns out to suffice to require that the sum of maximal row elements in $S(B \otimes J)$ is of order $O(\log N)$. And for this, it suffices w.h.p. that the sum of maximal row elements in $S(B)$ is of order $O(\log n)$. Using large deviations.
- Bad integral

\[ I_b = \int_{\text{bad } B} \text{Per}(B \otimes J) d(B) \]

is over bad matrices $B$ - with large sum of maximal elements.
- Turns out that $I_b$ counts bad tables in $\Sigma(n)$ - these with specific structure: large sum of maximal elements.
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is over bad matrices \( B \) - with large sum of maximal elements.
- Turns out that \( I_b \) counts bad tables in \( \Sigma(n) \) - these with specific structure: large sum of maximal elements.
- The number of such tables is \textbf{negligible} via a combinatorial argument.
Approximating the good integral

- We have

\[ |\Sigma(n)| \approx \int_{\text{good } B} \Per(B \otimes J) d(B) = \int_{\text{good } B} \sigma(B \otimes J) \cdot \Per(S(B \otimes J)) d(B) \]
Approximating the good integral

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- For good \( B \) holds: \( e^{-N} < \text{Per}(S(B \otimes J)) \leq N^{O(\log(N))} \cdot e^{-N} \).
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- Hence, up to a factor of $N^{O(\log(N))}$,

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- Can we compute this?
Approximating the good integral

- We have

\[ |\Sigma(n)| \approx \int_{\text{good } B} \text{Per}(B \otimes J) d(B) \]

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- Hence, up to a factor of \( N^{O(\log(N))} \),

\[ |\Sigma(n)| \approx e^{-N} \int_{\text{good } B} \sigma(B \otimes J) d(B) \]

- Can we compute this? Yes, since \( \sigma(B \otimes J) \) is log-concave in \( B \). B’05, G’06.
Log-concave functions

- A nonnegative function $f : \mathbb{R}^N \rightarrow \mathbb{R}$ is log-concave if

  \[ f(\lambda x + (1 - \lambda)y) \geq f^\lambda(x) \cdot f^{1-\lambda}(y) \]
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- Log-concave functions are easy to integrate within an arbitrary error AK ’91 (with application to computing volumes of convex bodies).
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- E.g., the characteristic function of a convex body is log-concave. And so is the exponential density
  \[f(B) = \exp\{-\sum b_{i,j}\}.
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- Log-concave functions are easy to integrate within an arbitrary error \( \text{AK '91} \) (with application to computing volumes of convex bodies).
- Hence \( \int_{\text{good } B} \sigma(B \otimes J)d(B) \) is computable.
Log-concave functions

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- Log-concave functions are easy to integrate within an arbitrary error AK ’91 (with application to computing volumes of convex bodies).

- Hence $\int_{\text{good } B} \sigma(B \otimes J)d(B)$ is computable. Done.
The general contingency tables partition function

Barvinok ’07–’16

- Use
The general contingency tables partition function
Barvinok ’07-’16

- Use
  - (Generalized) log-concavity of the scaling factor function.
The general contingency tables partition function
Barvinok ’07–’16

- Use
  - (Generalized) log-concavity of the scaling factor function,
  - Strong concentration of the permanent for random ‘exponential’ matrices.
The general contingency tables partition function

Barvinok ’07–’16

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  - (Generalized) log-concavity of the scaling factor function,
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  - The Prekopa-Leindler inequality.
The general contingency tables partition function
Barvinok ’07-’16

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  - (Generalized) log-concavity of the scaling factor function,
  - Strong concentration of the permanent for random ‘exponential’ matrices,
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- For a structural result: approximate log-concavity of the partition function.
The general contingency tables partition function
Barvinok ’07–’16

- A structural result: approximate log-concavity of the partition function.
- Recall the partition function with margins $R$ and $C$ and weights $W$:
The general contingency tables partition function
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$$p_{\Sigma(R,C)}(W) = p_{\Sigma(R,C)}(w_{11}, \ldots, w_{nn}) = \sum_{D=(d_{ij})} \prod_{i,j=1}^{n} w_{ij}^{d_{ij}}$$
• A structural result: approximate log-concavity of the partition function.

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• If $R = \sum_{i} \lambda_{i} R_{i}$ and $C = \sum_{i} \lambda_{i} C_{i}$ (a convex combination) then
The general contingency tables partition function
Barvinok ’07’16

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$$\rho_{\Sigma(R,C)}(W) = \rho_{\Sigma(R,C)}(w_{11}, \ldots, w_{nn}) = \sum_{D=(d_{ij})} \prod_{i,j=1}^{n} w_{ij}^{d_{ij}}$$

• If $R = \sum_i \lambda_i R_i$ and $C = \sum_i \lambda_i C_i$ (a convex combination) then

$$\rho_{\Sigma(R,C)}(W) \geq \beta \cdot \prod_i \rho_{\Sigma(R_i,C_i)}^\lambda(W)$$
The general contingency tables partition function

Barvinok '07-'16

- A structural result: approximate log-concavity of the partition function.
- Recall the partition function with margins $R$ and $C$ and weights $W$:

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where $\beta$ is not too small.
The general contingency tables partition function
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where $\beta$ is not too small.

- This and a subtle application of the capacity theory for polynomials (Gurvits) leads to approximation of general partition functions.
Convex optimization
Existence and properties of scaling factors

- Let $A - (a_{ij})$ be an $n \times n$ matrix with positive entries.
Let $A = (a_{ij})$ be an $n \times n$ matrix with positive entries. We know that there exist $\{\lambda_i\}$ and $\{\mu_j\}$ such that $S(A) = (\lambda_i a_{ij} \mu_j)$ is doubly stochastic.
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• We will give a new proof of this fact, with some nice side benefits, via convex optimization.
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The scaling factors $\{\lambda_i\}$ and $\{\mu_j\}$ can be found efficiently.
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  • The function $\sigma(A) = \left( \prod_i \lambda_i \cdot \prod_j \mu_j \right)^{-1}$ is log-concave.
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• This approach to scaling extends to other settings.
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  • The function $\sigma(A) = \left(\prod_i \lambda_i \cdot \prod_j \mu_j\right)^{-1}$ is log-concave.
  • This approach to scaling extends to other settings.

• The iterative approach of LSW’98 is strongly polynomial but seems to be limited to the matrix scaling case.
Scaling factors as a minimum of convex function

Following B’06

- Given $A$, let

$$f_A(x_1, \ldots, x_n) = \sum_{i=1}^{n} \ln \left( \sum_{j=1}^{n} a_{ij} e^{x_j} \right)$$
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- $f_A$ is strongly convex in $x = (x_1, \ldots, x_n)$. 
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• $f_A$ is strongly convex in $x = (x_1, \ldots, x_n)$. That is, for $0 < \lambda < 1$ holds $f(\lambda x^{(1)} + (1 - \lambda)x^{(2)}) < \lambda f(x^{(1)}) + (1 - \lambda)f(x^{(2)})$. 
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• In addition $f_A$ tends to infinity if at least one of the $x_i$ does.
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• In addition $f_A$ tends to infinity if at least one of the $x_i$ does.

• This means $f_A$ has a unique minimum $x^* = (x_1^*, \ldots, x_n^*)$ on $H = \{x_1 + \ldots + x_n = 0\}$. 
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- The first order optimality conditions for $x^*$ give
Scaling factors as a minimum of convex function
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- The first order optimality conditions for $x^*$ give

$$\frac{\partial f_A}{\partial x_k} = \sum_{i=1}^{n} \frac{a_{ik} e^{x_k^*}}{\sum_{j=1}^{n} a_{ij} e^{x_j^*}} = \gamma$$
Scaling factors as a minimum of convex function
Following B’06

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f_A(x_1, \ldots, x_n) = \sum_{i=1}^{n} \ln \left( \sum_{j=1}^{n} a_{ij}e^{x_j} \right)\]

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\]

for some constant \( \gamma \) and for all \( k \).
Scaling factors as a minimum of convex function
Following B’06

• $f_A(x_1, \ldots, x_n) = \sum_{i=1}^{n} \ln \left( \sum_{j=1}^{n} a_{ij} e^{x_j} \right)$.

• $x^* = \min_{x_1 + \ldots + x_n = 0} f_A(x)$.

• The first order optimality conditions for $x^*$ give

$$\frac{\partial f_A}{\partial x_k} = \sum_{i=1}^{n} \frac{a_{ik} e^{x^*_k}}{\sum_{j=1}^{n} a_{ij} e^{x^*_j}} = \gamma,$$

for some constant $\gamma$ and for all $k$. 
Scaling factors as a minimum of convex function

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- The first order optimality conditions for \( x^* \) give
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  \]
  for some constant \( \gamma \) and for all \( k \).
- Let \( \lambda_i = \left( \sum_{j=1}^{n} a_{ij} e^{x_j^*} \right)^{-1} \) and \( \mu_j = e^{x_j^*} \).
Scaling factors as a minimum of convex function
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- Each column sum in \( B \) is 1 and each row sum is \( \gamma \).
Scaling factors as a minimum of convex function
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  \]
  for some constant \( \gamma \) and for all \( k \).
- Let \( \lambda_i = \left( \sum_{j=1}^{n} a_{ij} e^{x_j^*} \right)^{-1} \) and \( \mu_j = e^{x_j^*}. \) Let \( B = (\lambda_i a_{ij} \mu_j). \)
- Each column sum in \( B \) is 1 and each row sum is \( \gamma \). Hence \( \gamma = 1 \) and \( B \) is doubly stochastic.
Scaling factors as a minimum of convex function
Following B’06

- $f_A(x_1, \ldots, x_n) = \sum_{i=1}^{n} \ln \left( \sum_{j=1}^{n} a_{ij} e^{x_j} \right)$.
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- \( B \) is doubly stochastic.
- The scaling factors \( \{\lambda_i\} \) and \( \{\mu_j\} \) can be found using the ellipsoid algorithm.
Scaling factors as a minimum of convex function

Following B’06

- $f_A(x_1, \ldots, x_n) = \sum_{i=1}^{n} \ln \left( \sum_{j=1}^{n} a_{ij} e^{x_j} \right)$.
- $x^* = \min_{x_1 + \ldots + x_n = 0} f_A(x)$.
- Let $\lambda_i = \left( \sum_{j=1}^{n} a_{ij} e^{x_j^*} \right)^{-1}$ and $\mu_j = e^{x_j^*}$. Let $B = (\lambda_i a_{ij} \mu_j)$.
- $B$ is doubly stochastic.
- The scaling factors $\{\lambda_i\}$ and $\{\mu_j\}$ can be found using the ellipsoid algorithm, attaining (arbitrary) precision $\epsilon$ in time $\text{Poly}(n, \log(1/\epsilon), \nu)$, where $\nu = \log \left( \max\{a_{ij}\} / \min\{a_{ij}\} \right)$. 
Log-concavity of $\sigma(A)$

Following B’06

- We have

$$\sigma(A) = \left( \prod_i \lambda_i \right)^{-1} \cdot \left( \prod_j \mu_j \right)^{-1}$$
Log-concavity of $\sigma(A)$

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- We have

$$\sigma(A) = \left( \prod_i \lambda_i \right)^{-1} \cdot \left( \prod_j \mu_j \right)^{-1} = \left( \prod_{i=1}^{n} \sum_{j=1}^{n} a_{ij} e^{x_j^*} \right) \cdot e^{-\sum_j x_j^*}$$
Log-concavity of $\sigma(A)$
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Log-concavity of $\sigma(A)$
Following B’06

- We have

$$\sigma(A) = \left( \prod \lambda_i \right)^{-1} \cdot \left( \prod \mu_j \right)^{-1} = \left( \prod \sum_{i=1}^{n} a_{ij} e^{x_j^*} \right) \cdot e^{-\sum_j x_j^*} =$$

$$\prod_{i=1}^{n} \sum_{j=1}^{n} a_{ij} e^{x_j^*} = e^{f_A(x^*)}$$

- Hence $\ln \sigma(A) = f_A(x^*) = \min_{x_1+\ldots+x_n=0} f_A(x)$. 
Log-concavity of $\sigma(A)$

Following B’06

- $\ln \sigma(A) = f_A(x^*) = \min_{x_1 + \ldots + x_n = 0} f_A(x)$.
- For a fixed $x$, $f_A(x) = \sum_{i=1}^n \ln \left( \sum_{j=1}^n a_{ij} e^{x_j} \right)$ is concave in $A$. 
Log-concavity of $\sigma(A)$
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- Minimizing for $x$:
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- Done.
• Permanents.
• Contingency tables and integer flows.
• Mixed discriminants.
Higher-dimensional partition functions - mixed discriminants

- Let $A_1, \ldots, A_n$ be symmetric $n \times n$ matrices.
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- The best known approximation factor is $n^{O(n)}$ B’97.
Deterministic approximation of mixed discriminant and mixed volume

- **DGH’98**: Is there a deterministic algorithm to approximate mixed volume?
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- **GS’01**: A deterministic algorithm to approximate mixed discriminants (of psd matrices) up to a factor of \(e^n\) using scaling.
Scaling $n$-tuples of matrices

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- Scaling to what? Need to define a goal object, that will be sufficiently balanced so that mixed discriminant is concentrated.
- The second step will require proving upper and lower bounds on the mixed discriminant.
Allowed scaling operations

- \( D(A_1, \ldots, A_n) = \sum_{\sigma, \tau \in S_n} \text{sign}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}^{(\tau(i))} \) is multi-linear, so "row" and "column" scaling should work.
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- In this way mixed discriminant is a non-commutative generalization of permanent.
Scaling to a doubly stochastic tuple

- An $n$-tuple $B_1, \ldots, B_n$ of psd matrices is doubly stochastic if
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- The commutative case corresponds to a usual doubly stochastic matrix.
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  • The scaling factors can be found efficiently.
Proof - via convex optimization

- Let $A = \{A_i\}$ be a positive $n$-tuple of psd matrices.
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- $f_A$ is strongly convex in $x = (x_1, \ldots, x_n)$. 
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• They can be found efficiently (but not strongly polynomially) using the ellipsoid method.
Concentration of mixed discriminant for doubly stochastic tuples

- For any $n$-tuple $A = A_1, ..., A_n$ of psd matrices
  $D(A) = D(A_1, ..., A_n) \leq \text{Det}(A_1 + ... A_n)$. 
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• Following this, Leonid Gurvits developed a new general approach to lower bound coefficients of \( H \)-stable polynomials via their capacity. This led to easy proofs for the permanent and mixed discriminant lower bounds and other results.
Rado’s theorem with volume
A geometric corollary

- $V_1, ..., V_n$ are families of vectors in $\mathbb{R}^n$. 
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- $V_1, \ldots, V_n$ are families of vectors in $\mathbb{R}^n$. Assume for all $S \subseteq [n]$ there are $|S|$ independent vectors $v_1^{(S)}, \ldots, v_{|S|}^{(S)}$ in $\bigcup_{i \in S} V_i$. 

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Operator scaling

- Let $A_1, \ldots, A_m$ be $m \times n \times n$ matrices. Do there exist matrices (scaling factors) $X$ and $Y$ such that for $B_i = XA_iY$ holds
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Thank you!