Derivative pricing with linear volatility models

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Affine models for derivatives pricing

Affine models are widely used:

- In term structure models: Vasicek (1977), Cox, Ingersoll and Ross (1985), Duffie and Kan (1996), Chen (1996), Balduzzi et al. (1996), Dai and Singleton (2000), ...


- For dynamic asset allocation: Liu (2007)

- Other areas include RV modelling, pricing of credit derivatives, commodity prices etc.
Affine models for derivatives pricing

They enjoy many desirable features:

- Joint characteristic function of log price and latent variables is exponential affine in the variables.
- Allows to derive a "closed-form" solution for option / zero-coupon bond prices using Fourier inversion.
- Estimation of time-series is possible, though cumbersome.
- Risk premia are available in closed-form.
Non-affine versus affine models

The square root model has been shown not to fit return data very well:

- The term structure of volatility is not well represented (Pan, 2002).
- The square root model generates insufficient kurtosis for the returns (Andersen, Benzoni and Lund, 2002).
- The distribution of the volatility of returns is closer to an inverse gamma distribution than to a gamma distribution (Bouchaud and Potters, 2003).

Jumps have become the standard tool to overcome these limitations. But (Bakshi, Cao and Chen, 1997, 2000) argue that

- They have little effect on pricing and hedging of long-term option prices.
- Their hedging performance is worse than without jumps for short maturities.
- Concern of overfitting.
Jumps or vol-of-vol?

Periods of high volatility coincide with periods of volatile volatility (not in line with the square root model).
Jumps or vol-of-vol?

Recent literature on high-frequency data questions the role of jumps (Christensen, Oomen and Podolskij, 2014).
Beyond affine models

Let $V_t$ be the diffusive variance of returns and $v_t = \sqrt{V_t}$.

- **GARCH diffusion model** (extends to the power ARCH: Fornari and Mele, 2001).

  $$dV_t = \kappa (\theta - V_t) dt + \lambda V_t dB_t$$

  Has been found to fit S&P 500 and VIX option prices and returns better than the Heston model (Christoffersen, Jacobs and Mimouni, 2010).

- **Double lognormal stochastic volatility model** (Gatheral, 2007, 2008; Henry-Labordère, 2009)

  $$dV_t = \kappa (V'_t - V_t) dt + \lambda V_t dB_t$$

  $$dV'_t = \kappa' (\theta - V'_t) dt + \lambda' V'_t dB'_t$$

  Has been found to fit S&P 500 and VIX option prices better than the double Heston model (Gatheral, 2008).

- **Generalised Inverse Gamma model** (Ma and Serota, 2014)

  $$dv_t = \kappa (\theta v_t^{1-\gamma} - v_t) dt + \lambda v_t dB_t$$
Beyond affine models

- Lognormal Beta model (Sepp, 2014 and 2015)
  \[ dv_t = \kappa (\theta - v_t) dt + \beta v_t dW_t + \epsilon v_t dB_t \]

- Inverse Gamma model (Langrené, Lee and Zhu, 2016)
  \[ dv_t = \kappa (\theta - v_t) dt + \lambda v_t dB_t \]
Beyond affine models

One step further:

- CEV model (Chan et al., 1992; Lewis, 2000; Chacko and Viceira, 2001)
  \[ dV_t = \kappa (\theta V_t - V_t^2) dt + \lambda V_t^\gamma dB_t \]
  Moments of order \( > 1 \) may not exist. No closed-form expression for the option price.

- Special case of the 3/2 model
  \[ dV_t = \kappa (\theta V_t - V_t^2) dt + \lambda V_t^{3/2} dB_t \]
  Quadratic volatility of volatility. But has been found to fit option prices and returns less well than the GARCH diffusion model (Christoffersen, Jacobs and Mimouni, 2010).

In fact, Christoffersen, Jacobs and Mimouni (2010) find, using realized volatility, VIX index, S&P 500 returns and S&P 500 (out-of-the-money) option prices, that "the best volatility specification is one with linear rather than square root diffusion for variance".

They use Monte-Carlo simulations to price options.
⇒ Daily log changes in realized volatility almost have a normal distribution, as in linear volatility models.
Q-Q. plots of VIX

⇒ A bit less clear with the VIX.
Beyond affine models

Other non-affine models have been proposed

- So that volatility of volatility be driven by a separate factor: unspanned vol-of-vol

\[
\begin{align*}
    dV_t &= \kappa_v (m_t - V_t) dt + \sigma_v \sqrt{q_t} dW_t^v + Z_t^v dN_t^v \\
    dm_t &= \kappa_m (\bar{m} - m_t) dt + \sigma_m \sqrt{m_t} dW_t^m \\
    dq_t &= \kappa_q (\bar{q} - q_t) dt + \sigma_q \sqrt{q_t} dW_t^q
\end{align*}
\]

(Branger, Kraftschik and Völkert, 2016) estimate the model using the VIX index, VIX futures and VIX options. They argue that the additional factor improves the fit to VIX derivatives and allows to better represent the risk-neutral moments of the VIX.
Beyond affine models

- To allow for quadratic variance: (Filipović, G. and Mancini, 2016)

\[
V_t = \pi_0 + \pi_1^T X_t + X_t^T \pi_2 X_t \\
dX_t = \mu(X_t)dt + \Sigma(X_t)dW_t;
\]

We show that this model fits variance swap rates slightly better than an affine model with jumps.

- To allow for downward volatility jumps: (Amengual and Xiu, 2016)

\[
V_t = \pi_0 + \pi_1^T X_t + X_t^T \pi_2 X_t + \exp(\pi_3 + \pi_4^T X_t)
\]

with \(X_t\) affine jump diffusion. They show that the model fits variance swaps and S&P 500 returns better than affine models.
Growth of derivatives market

**S&P 500 (SPX) Options**
Average Daily Volume rose to 1.16 million in Jan.-Apr. 2017

**VIX® Options**
Avg. Daily Volume rose to 712,490 in Jan.-Apr. 2017
$100 Multiplier

**VIX® Futures**
Average Daily Volume rose to 265,954 in Jan.-Apr. 2017
$1000 multiplier

Source: CBOE (includes Weeklys and EOM) [www.cboe.com/SPX](http://www.cboe.com/SPX)

Source: CBOE Volatility Index® (VIX®) Source: CBOE [www.cboe.com/VIX](http://www.cboe.com/VIX)

Source: CBOE Holdings [www.cboe.com/VIX](http://www.cboe.com/VIX)
Put options are mostly traded on the S&P 500, and call options on the VIX. An example of implied volatility smile, on May 10th, 2010:
Contribution

This paper:

- Studies a range of valuation problems in the setting of linear volatility models
- Describes how to price S&P 500 and VIX options
- Examines the added value of stochastic volatility of volatility and unspanned volatility.
Motivation

Linear volatility models

Outline

1. Linear volatility models

2. Estimation

3. Estimation results

Derivative pricing with linear volatility models
General specification

- Filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{Q})\).
- \(S_t\) be a semimartingale of the form
  \[
  \frac{dS_t}{S_t^-} = r_t \ dt + \sigma_t \ dB_t + \int_{\mathbb{R}} \xi (\chi(dt, d\xi) - \nu_t(d\xi)dt),
  \]
- Let \(X_t\) be a state process in \(\mathbb{R}^d\) satisfying
  \[
  dX_t = (b + \beta X_t) \ dt + \sum_{j=1}^{m} (c_j + \gamma_j X_t) \ dW_{j,t},
  \]
- Set \(\sigma_t = |X_{1t}|\). The spot variance is given by
  \[
  V_t = X_{1t}^2 + \int_{\mathbb{R}} (\log(1 + \xi))^2 \nu_t(d\xi).
  \]
- The diffusive variance follows
  \[
  d\sigma_t^2 = \left[ 2\sigma_t (b_1 + \beta_1 X_t) + \sum_{j=1}^{m} (c_{j,1} + \gamma_{j,1} X_t)^2 \right] dt + 2\sigma_t \sum_{j=1}^{m} (c_{j,1} + \gamma_{j,1} X_t) dW_{j,t},
  \]
The leverage effect is represented by setting $dB_t = \ell^T dW_t$.

$$
\text{Corr} \left( \frac{dS_t}{S_t}, d\sigma_t^2 \right) = \frac{\sum_{j=1}^m \ell_j (c_{j,1} + \gamma_{j,1}X_t)}{\sqrt{\sum_{j=1}^m (c_{j,1} + \gamma_{j,1}X_t)^2}},
$$

The process $Z_t = (X_{1,t}, \ldots, X_{d,t}, L_t)^T$, with $L_t = \ln S_t$ is not a polynomial jump-diffusion (PJD) process $\Rightarrow$ Need to augment it (Filipović and Larsson, 2016).
General specification

\[ \tilde{Z}_t = (X_{1,t}, ..., X_{d,t}, X_{1,t}^2, ..., X_{d,t}^2, L_t)^\top \] has generator

\[
\mathcal{G} f(z) = a(x)^\top \nabla f(z) + \frac{1}{2} \sum_{i,j=1}^{2d+1} A_{ij}(x) \frac{\partial^2 f}{\partial z_i \partial z_j}(z) \\
+ \int_{\mathbb{R}} (f(z + \log(1 + \xi) e_{2d+1}) - f(z)) \, \nu(x, d\xi),
\]

where \( a(x) \in Pol_2(\mathbb{R}^d) \) and \( A_{ij}(x) \in Pol_4(\mathbb{R}^d) \). Assume \( \nu(x, d\xi) = \lambda x_i^2 d\xi \). \( \tilde{Z}_t \) is a PJD and therefore has the polynomial preserving property.

Motivation

Linear volatility models

Estimation

Estimation results

Closed-form moments

Moments of $X$ and $L$

Let $D = \sum_{k=0}^{N} \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{k-2j+d-1}{k-2j}$ denote the dimension of the space of polynomials in $\bar{Z}_T$ of degree $N$ or less. The $D$-row vector of the mixed $\mathcal{F}_t$-risk-neutral conditional moments of $\bar{Z}_T$ of order $N$ or less with $T \geq t$ is given by

$$
\begin{pmatrix}
1, \mathbb{E}_t[X_{1,T}], \ldots, \mathbb{E}_t[X_{d-1,T} X_1^{N-1}, \ldots, \mathbb{E}_t[X_{d,T}^{-2} L_T], \ldots, \mathbb{E}_t \left[ L_T^{\left\lfloor \frac{N}{2} \right\rfloor} X_{d,T}^{2\left(\frac{N}{2} - \left\lfloor \frac{N}{2} \right\rfloor\right)} \right]
\end{pmatrix}
$$

$$
= \begin{pmatrix}
1, \ldots, X_{d-1,t} X_1^{N-1}, \ldots, X_{d,t}^{N-2} L_t, \ldots, L_t^{\left\lfloor \frac{N}{2} \right\rfloor} X_{d,t}^{2\left(\frac{N}{2} - \left\lfloor \frac{N}{2} \right\rfloor\right)}
\end{pmatrix} e^{B(T-t)},
$$

Denote $H_T^T = \begin{pmatrix}
1, \ldots, X_{d-1,t} X_1^{N-1}, \ldots, X_{d,t}^{N-2} L_t, \ldots, L_t^{\left\lfloor \frac{N}{2} \right\rfloor} X_{d,t}^{2\left(\frac{N}{2} - \left\lfloor \frac{N}{2} \right\rfloor\right)}
\end{pmatrix}$.
Pricing of stock options

- The characteristic function of $L_T|\mathcal{F}_{t_0}$ can be expanded as follows:

$$\mathbb{E}_Q^\mathcal{F}_{t_0}[e^{zL_T}] = \exp\left(\sum_{k=1}^{\infty} \frac{C_n z^n}{n!}\right) = \exp\left(C_1 z + \frac{C_2 z^2}{2}\right) \left(1 + C_3 \frac{z^3}{3!} + O(z^4)\right),$$

where the cumulants $C_i$ are obtained from the moments.

- Fourier inversion can be used to recover the option price.
Pricing of volatility instruments

The VIX squared is defined as a finite sum of calls and puts that converges when there exists calls and puts for all strikes $K \in \mathbb{R}_+$ to the integral

$$\text{VIX}_t^2 \approx \frac{1}{\tau} \mathbb{E}_t \left[ \int_t^{t+\tau} \sigma_u^2 du + 2 \int_t^{t+\tau} \int_{\mathbb{R}} (\xi - \ln(1 + \xi)) \nu_t(d\xi)du \right]$$

Similarly, the variance swap rate is given by

$$\text{VS}(t, t+\tau) = \frac{1}{\tau} \mathbb{E}_t \left[ \int_t^{t+\tau} \sigma_u^2 du + \int_t^{t+\tau} \int_{\mathbb{R}} \ln(1 + \xi)^2 \nu_u(d\xi)du \right]$$

With deterministic jumps of size $\xi$, the calculations of the VIX and VS rate boil down to calculating

$$\mathbb{E}_t \left[ \int_t^{t+\tau} \sigma_u^2 du \right] = \mathbb{E}_t \left[ \int_t^{t+\tau} X_{1,u}^2 du \right] = \int_t^{t+\tau} \mathbb{E}_t \left[ X_{1,u}^2 du \right]$$

$$= \int_t^{t+\tau} H_t^T \cdot e^{B(u-t)} e_j du = H_t^T \cdot \int_t^{t+\tau} e^{B(u-t)} e_j du.$$

where $j$ is such that $H_t(j) = X_{1,t}^2$. 
### Pricing of VIX options

Assuming deterministic jump sizes, moments of the VIX square are given by:

\[
\mathbb{E}_t \left[ (VIX^2_T)^k \right] = \mathbb{E}_t \left[ \left( 1 + 2(\xi - \ln(1 + \xi))\lambda \right)^{\tau \mathbb{E}_T} \left[ \int_T^{T+\tau} \mathcal{X}_{1,s}^2 ds \right] \right]^k
\]

\[
= C_{VIX}^k \mathbb{E}_t \left[ \left( \int_T^{T+\tau} H_T^T \cdot e^{B(s-T)} e_j ds \right)^k \right] = C_{VIX}^k \mathbb{E}_t \left[ \left( H_T^T \cdot \int_T^{T+\tau} e^{B(s-T)} e_j ds \right)^k \right]
\]

\[
= C_{VIX}^k \mathbb{E}_t \left[ \left( H_T^T \cdot \int_0^T e^{B_s} e_j ds \right)^k \right]
\]

Similarly to S&P 500 options, the characteristic function of the VIX square is computed using an Edgeworth expansion and VIX option prices are recovered by polynomial expansion (Filipović, Mayerhofer and Schneider, 2013; Ackerer, Filipović and Pulido, 2016) or Fourier inversion (Fang and Osterlee, 2008).
Three special cases of linear volatility models

**Univariate model:** GARCH-type

\[ dX_t = \kappa(\theta - X_t)dt + (c + \gamma X_t)dW_t. \]

**Bivariate model:** stochastic central tendency

\[ dX_{1,t} = \kappa_1(X_{2,t} - X_{1,t})dt + (c_1 + \gamma_1 X_{1,t})dW_{1,t}, \]
\[ dX_{2,t} = \kappa_2(\theta - X_{2,t})dt + (c_2 + \gamma_2 X_{2,t})dW_{2,t}, \]

**Trivariate model:** unspanned vol-of-vol

\[ dX_{1,t} = \kappa_1(X_{2,t} - X_{1,t})dt + (\gamma_1 X_{1,t} + X_{3,t})dW_{1,t}, \]
\[ dX_{2,t} = \kappa_2(\theta_2 - X_{2,t})dt + (c_2 + \gamma_2 X_{2,t})dW_{2,t}, \]
\[ dX_{3,t} = \kappa_3(\theta_3 - X_{3,t})dt + (c_3 + \gamma_3 X_{3,t})dW_{3,t}. \]
Outline

1. Linear volatility models
2. Estimation
3. Estimation results
Data on S&P 500 and VIX markets:

- Daily SPX and VIX index values from March 2006 to end of October 2010
- Daily realized variances computed from high-frequency data
- Prices of S&P 500 options: maturities from 6 days to 1 year, moneyness $K/F$ from 0.4 to 1.35
- Prices of VIX options: maturities from 6 days to 6 months, moneyness $K/F$ from 0.55 to 2

In-sample period: March 1st, 2006 until end of November, 2008.
Out-of-sample period: December, 2008 until end of October, 2010
## Cross-section of options

<table>
<thead>
<tr>
<th>SPX options</th>
<th>In-sample</th>
<th>Out-of-sample</th>
</tr>
</thead>
<tbody>
<tr>
<td>Overall</td>
<td>14351</td>
<td>13323</td>
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<tr>
<td>Deep OTM puts</td>
<td>1365</td>
<td>3046</td>
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<tr>
<td>OTM puts</td>
<td>6498</td>
<td>4755</td>
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<td>Close to ATM options</td>
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<td>OTM calls</td>
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<td>Short mat options</td>
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<td>Medium mat options</td>
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<td>5678</td>
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<td>Long mat options</td>
<td>4376</td>
<td>4065</td>
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<table>
<thead>
<tr>
<th>VIX options</th>
<th>In-sample</th>
<th>Out-of-sample</th>
</tr>
</thead>
<tbody>
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<td>Overall</td>
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<td>5118</td>
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<tr>
<td>Deep OTM calls</td>
<td>1100</td>
<td>2039</td>
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<tr>
<td>OTM calls</td>
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<td>916</td>
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<tr>
<td>Close to ATM options</td>
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<td>1109</td>
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<tr>
<td>ITM calls</td>
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<td>1054</td>
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<tr>
<td>Short mat options</td>
<td>1948</td>
<td>1054</td>
</tr>
<tr>
<td>Medium mat options</td>
<td>2200</td>
<td>2237</td>
</tr>
</tbody>
</table>
Option pricing

Measurements considered:

- VIX levels: $\epsilon_t^{VIX^2} \sim \mathcal{N}(0, s)$

- Options

\[
O_{t,i}^{SPX,Mod}(Y_t, \nu_t, m_t, \Theta^Q, \Theta^P, Q) - O_{t,i}^{SPX,Mkt} \sim \mathcal{N}(0, \sigma_{\epsilon_{t,i}^{SPX}}^2) \quad (1)
\]

\[
C_{t,j}^{VIX,Mod}(\nu_t, m_t, \Theta^Q, \Theta^P, Q) - C_{t,j}^{VIX,Mkt} \sim \mathcal{N}(0, \sigma_{\epsilon_{t,j}^{VIX}}^2) \quad (2)
\]
Unscented Kalman Filter

Algorithm: Unscented Kalman Filter (Julier, 1999; Julier et al., 2002)

- Time-consistent estimation.
- Very fast.

Assumptions made:

- VIX levels and option prices observed with error.
- Option errors independent from one another at time $t$, with heteroskedastic variances.

Outputs:

- At each $t$, estimate of the latent factors.
- At each $t$, likelihood of observations up to $t$: $p(\bar{y}_t; \Theta)$. 
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Estimation results

Univariate model

Fitting of VIX spot values

Market data

Filtered trajectory

Scatterplot of VIX data versus fitted points

Sequential log-likelihood

Derivative pricing with linear volatility models
Bivariate model

Fitting of VIX spot values

Scatterplot of VIX data versus fitted points

Sequential log-likelihood

Derivative pricing with linear volatility models
Trivariate model

Sequential log-likelihood

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Derivative pricing with linear volatility models
Future work

- Add forward log-returns and realized variance in the measurement equations to estimate jointly the $\mathbb{P}$ parameters.

- Examine whether the departure from affine models affects risk premia.
Conclusion

- Linear stochastic volatility models allow overcoming many restrictions of affine models while keeping their tractability.

- In particular, they allow for a more flexible representation of the volatility of volatility.

- They capture large spikes in the VIX, which is difficult with a diffusive affine model.