Libor and forward process models from an affine point of view
Another didactic note

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Outline

1. Interest rate models
2. Affine processes and integration
3. Affine models: factor, HJM, exponential-rational, linear-rational
4. LIBOR models: the classical (or lognormal) point of view
5. LIBOR models: the forward process (or affine) point of view
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Interest rate models
– from a basic structural point of view

Desirable properties: tractability and flexibility

Objects of interest:

- zero coupon bonds $B(t, T)$ with $B(T, T) = 1$
- bond options with time-$T_0$-payoff $f(B(T_0, T_1))$ with $T_0 < T_1$
- caps/floors with caplet/floorlet time-$T_n$-payoffs
  $(L(T_{n-1}, T_{n-1}, T_n) - \ell)^+ \Delta T$ resp. $(\ell - L(T_{n-1}, T_{n-1}, T_n))^+ \Delta T$ for $L(t, T_{n-1}, T_n) := \frac{B(t, T_{n-1}) - B(t, T_n)}{B(t, T_n) \Delta T}$
- swaptions with time-$T_0$-payoff
  $(1 + B(T_0, T_N) - \sum_{n=1}^N B(T_0, T_n) \ell \Delta T)^+$
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Interest rate models
– from a basic structural point of view, ct’d

- Caplets/floorlets reduce to put/call options on the bond:

\[
\text{Value}\left(\text{caplet with time-}T_n\text{-payoff} \ (L(T_{n-1}, T_{n-1}, T_n) - \ell)^+ \Delta T\right) \\
= \text{Value}\left(\text{time-}T_n\text{-payoff} \ (1 - (1 + \ell \Delta T)B(T_{n-1}, T_n))^+\right)
\]

- Swaptions sometimes reduce to put options on the bond:

  - consider time-\(T_0\)-payoff \(H = (K - \sum_{n=1}^{N} c_n B(T_0, T_n))^+\)
  - suppose \(B(T_0, T_n) = \pi(T_0, T_n, X(T_0))\) for some univariate process \(X\), strictly decreasing functions \(\pi(T_0, T_n, \cdot)\)
  - let \(x_0\) be solution to \(K - \sum_{n=1}^{N} c_n \pi(T_0, T_n, x) = 0\)
  - set \(K_n := \pi(T_0, T_n, x_0)\) for \(n = 1, \ldots, N\)
  - Jamshidian’s representation: \(H = \sum_{n=1}^{N} c_n (K_n - B(T_0, T_n))^+\)
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Value(caplet with time-\(T_n\)-payoff \((L(T_{n-1}, T_{n-1}, T_n) - \ell)^+ \Delta T\))

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(Time-inhomogeneous) affine processes
– a tractable and flexible class of processes
(Duffie, Filipović, Schachermayer 2001, Filipović 2005, etc.)

- Affine local exponent or symbol:
  \[ q(t, u) := \psi_0(t, u) + \sum_{i=1}^{d} \psi_j(t, u)X_j(t-) \], i.e.
  \[ M(t) := e^{iu^\top X(t)} - \int_0^t e^{iu^\top X(s-)} q(s, u)ds, \; u \in \mathbb{R}^d \] local martingale

- Exponentially affine characteristic function:
  \[ E(\exp(iu^\top X(s + t)) | \mathcal{F}_s) = \exp(\psi_0(t, u) + \sum_{j=1}^{d} \psi_j(t, u)X_j(s)) \],
  where \( \psi_1, \ldots, d = (\psi_1, \ldots, \psi_d) \) and \( \psi_0 \) solve the ODE system
  \[ \frac{d}{dt} \psi_j(t, u) = \psi_j(t, -i\psi_1, \ldots, d(t, u)), \quad \psi_1, \ldots, d(0, u) = iu \]
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Affine processes
– a tractable and flexible class of processes, ct’d

- Stability under integration:
  \( \tilde{X} := (X_1, \ldots, X_d, X_{d+1}) \) affine for \( X_{d+1}(t) := \int_0^t X_d(s) \, ds \)

- Stability under “affine” measure changes:
  If \( Q \overset{loc}{\sim} P \) with density process \( Z = \mathcal{E}(X_d) \) or \( Z = \exp(X_d) \),
  \( X \) is affine under \( Q \) as well.

- Joint moments of \( (X(t_1), \ldots, X(t_n)) \):
  also explicit in terms of \( (\Psi_0, \ldots, \Psi_d) \)

- Generalised moments by integral representation:
  \[
  E(f(X(s + t))|\mathcal{F}_s) = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} E(e^{zX(s+t)}|\mathcal{F}_s)\tilde{f}(z) \, dz
  \]
  with bilateral Laplace transform
  \( \tilde{f}(z) := \int_{-\infty}^{\infty} f(x)e^{-zx} \, dx \)
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  E(f(X(s+t)) | \mathcal{F}_s) = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} E(e^{zX(s+t)} | \mathcal{F}_s) \tilde{f}(z) \, dz
  \]
  with bilateral Laplace transform
  \[
  \tilde{f}(z) := \int_{-\infty}^{\infty} f(x) e^{-zx} \, dx
  \]
Outline

1. Interest rate models
2. Affine processes and integration
3. Affine models: factor, HJM, exponential-rational, linear-rational
4. LIBOR models: the classical (or lognormal) point of view
5. LIBOR models: the forward process (or affine) point of view
Short rate or factor models
– based on affine processes

- Numeraire: money market account $S_0(t) = \exp(\int_0^t r(s)ds)$
- Short rate $r(t)$:
  component of $\mathbb{R}^d$-valued (time-inhomogeneous) affine process
  (under risk-neutral measure $Q_0$ for $S_0$)
- Bonds: $B(t, T) = \exp(\psi_0(t, T - t, -i) + \psi_X(t, T - t, -i)^\top X(t))$
  where $J(t) := \int_0^t r(s)ds,$

$$E_{Q_0}(\exp(iu^\top X(s + t) - J(s + t))|\mathcal{F}_s) = \exp(\psi_0(s, t, u) + \psi_X(s, t, u)^\top X(t) - J(s))$$
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– based on affine processes, ct’d

- Bond options with time-$T_0$-payoff $H = (B(T_0, T_1) - K)^+$:
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$$\pi_{T_0, T_1}(t, z) = \exp \left( -iz\psi_0(T_0, T_1 - T_0; -i) 
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- Swaptions (in one-factor case $X = r$):
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- Numeraire: money market account $S_0(t) = \exp(\int_0^t r(s)ds)$
- Forward rates: $df(t, T) = - \frac{d}{dt} \log B(t, T)$
- Forward rate dynamics:
  $$df(t, T) = \alpha(t, T)dt + \sigma(t, T)dX(t),$$
  where $\sigma(t, T)$ deterministic,
  $X \mathbb{R}^d$-valued (time-inhomogeneous) affine process
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- HJM drift condition:
  $$A(t, T) = -\psi_0(t, -i\Sigma(t, T)) + \sum_{m=1}^d \psi_m(t, -i\Sigma(t, T))X_m(t-),$$
  for $A(t, T) := \int_t^T \alpha(t, s)ds$, $\Sigma(t, T) := \int_t^T \sigma(t, s)ds$,
  $\psi_0, \ldots, \psi_d$ exponents corresponding to $X$
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- Bond options with time-$T_0$-payoff $H = (B(T_0, T_1) - K)^+$:
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  $\pi_{T_0, T_1}(t, z) = \exp\left(\psi_0(t, T_0, T_1, iz) + \psi_X(t, T_0, T_1, iz)^T X(t)\right) \frac{B(t, T_1)^z}{B(t, T_0)^{z-1}}$

- Swaptions:
  - suppose $X$ univariate (time-inhomogeneous) Lévy,
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– based on affine processes (Flesaker, Hughston 1996, Rogers 1997, etc.)

- No risk neutral measure for specific numeriare: use state price density process $Z$ instead, i.e. $SZ$ martingale for any liquid $S$
- Model for state price density process $Z$:
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- Implied short rate:
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  where $X$ some $\mathbb{R}^d$-valued (time-inhomogeneous) affine process such that $e^{X_d(t)}$ martingale,
  $a, b$ increasing deterministic functions

- Implied short rate: $r(t) = \frac{a'(t)+b'(t)e^{X_d(t)}}{a(t)+b(t)e^{X_d(t)}}$

- Bonds: $B(t, T) = \frac{a(T)+b(T)e^{X_d(t)}}{a(t)+b(t)e^{X_d(t)}}$
Exponential-rational models
– based on affine processes (Flesaker, Hughston 1996, Rogers 1997, etc.)

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Exponential-rational models
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Bond options and swaptions (without Jamshidian’s trick): consider time-$T_0$-payoff $H = (K - \sum_{n=1}^{N} c_n B(T_0, T_n))^+$:

value $\pi(t) = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} \pi(t, z) \frac{e^{zX_d(t)} \tilde{K}^{1-z}}{z(z-1)} \, dz \frac{\sum_{n=1}^{N} c_n b(T_n) - Kb(T_0)}{a(t)+b(t)e^{X_d(t)}}$,

where

$\pi(t, z) := \exp(\psi_0(t, T_0 - t, -iz) + \psi_X(t, T_0 - t, -iz)^{\top} X(t))$,

$\tilde{K} := \frac{Ka(T_0) - \sum_{n=1}^{N} c_n a(T_n)}{\sum_{n=1}^{N} c_n b(T_n) - Kb(T_0)}$,

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$= \exp(\psi_0(s, t, u) + \psi_X(s, t, u)^{\top} X(s) + iuX_d(s))$
Linear-rational models
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- Model for state price density process $Z$:
  \[ Z(t) = e^{-\alpha t}(\varphi + X_d(t)), \]
  where $X$ some $\mathbb{R}^d$-valued (time-inhomogeneous) affine process such that
  \[ dX(t) = (\beta^{(0)} + \beta X(t))dt + \text{martingale} \]
  with some $\alpha, \varphi \in \mathbb{R}$, $\beta^{(0)} \in \mathbb{R}^d$ and $\beta \in \mathbb{R}^{d \times d}$

- Implied short rate:
  \[ r(t) = \frac{(\beta^{(0)} + \beta X(t))d}{\varphi + X_d(t)} \]

- Bonds:
  \[ B(t, T) = e^{-\alpha(T-t)} \cdot \frac{e^{\beta(T-t)} \int_0^{T-t} e^{-\beta s} \beta^{(0)} ds + e^{\beta(T-t)} X(t)}{\varphi + X_d(t)} \]
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$u := \left( \sum_{n=1}^{N} c_n e^{-\alpha T_n} (e^{\beta(T_n-T_0)})_{dj} \right)_{j=1,...,d}$,
$\tilde{K} := K\varphi - \sum_{n=1}^{N} c_n e^{-\alpha T_n} (\varphi + (e^{\beta(T_n-T_0)} \int_{T_n-T_0}^{T_n-T_0} e^{-\beta s} \beta(0) ds)_{d})$, 

$$E(\exp(iu^\top X_d(s + t))|\mathcal{F}_s) = \exp(\psi_0(s, t, u) + \psi_X(s, t, u)^\top X(s) + iuX_d(s))$$
Outline

1. Interest rate models
2. Affine processes and integration
3. Affine models: factor, HJM, exponential-rational, linear-rational
4. LIBOR models: the classical (or lognormal) point of view
5. LIBOR models: the forward process (or affine) point of view
Classical LIBOR modelling
lognormal case and Lévy extension (Brace, Gątarek, Musiela 1997, Miltersen, Sandmann, Sondermann 1997, Eberlein, Özkan 2005, etc.)

- Numeriare: $T_n$-bond, $n = 1, 2, \ldots$
- LIBOR rates: $L(t, S, T) := \frac{B(t, S) - B(t, T)}{B(t, T)(T - S)}$
- LIBOR dynamics:
  \[ dL(t, T_{n-1}, T_n) = L(t, T_{n-1}, T_n)\sigma_i(t)dW(t), \]
  $\sigma(t)$ deterministic function,
  $W$ Wiener process under risk-neutral measure $Q_n$ for $B(\cdot, T_n)$
- Short rate: not specified
- Bonds: state variables
- Caplets with time-$T_n$-payoff: $H = (L(T_{n-1}, T_{n-1}, T_n) - \ell)^+ \Delta T$:
  value $\pi(t) = B(t, T_n)E_{Q_n}((L(T_{n-1}, T_{n-1}, T_n) - \ell)^+|\mathcal{F}_t) \Delta T$,
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- Swaptions: no nice formula (need another LIBOR model)
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Forward process models
– based on affine processes (Eberlein, Özkan 2005, etc.)

- Contin. comp. forward rate: \( f(t, S, T) := \frac{1}{T-S} \log \frac{B(t, S)}{B(t, T)} \)
- Numeraire: discrete money market account
\[
\tilde{S}_0(t) = \exp \left( - \sum_{m=1}^n f(T_{m-1}, T_{m-1}, T_m)(T_m - T_{m-1}) \right.
- \left. \tilde{f}(t, t, T_n)(T_n - t) \right) \quad \text{for } T_{n-1} \leq t < T_n
\]

- Forward process dynamics:
\[ df(t, T_{n-1}, T_n) = \alpha_n(t)dt + \sigma_n(t)dX(t), \]
where \(\sigma_n\) deterministic,
\(X \in \mathbb{R}^d\)-valued (time-inhomogeneous) affine process
(under risk-neutral measure \(\tilde{Q}_0\) for \(\tilde{S}_0\))
- Forward process drift condition:
\[
A_n(t) = \psi_0(t, -i\Sigma_n(t)) + \sum_{m=1}^d \psi_m(t, -i\Sigma_n(t))X_m(t) \quad \text{for } A_n(t) := \sum_{m=n(t)+1}^n (T_m - T_{m-1})\alpha_m(t),
\Sigma_n(t) := \sum_{m=n(t)+1}^n (T_m - T_{m-1})\sigma_m(t), \quad T_{n(t)-1} \leq t < T_{n(t)},
\psi_0, \ldots, \psi_d \) exponents corresponding to \(X\)
- Discrete short rate dynamics: is not fully determined by model
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\]

- Forward process dynamics:
  \( df(t, T_{n-1}, T_{n}) = \alpha_n(t)dt + \sigma_n(t)dX(t) \),
  where \( \sigma_n \) deterministic,
  \( X \in \mathbb{R}^d \)-valued (time-inhomogeneous) affine process
  (under risk-neutral measure \( \tilde{Q}_0 \) for \( \tilde{S}_0 \))
- Forward process drift condition:
  \( A_n(t) = \psi_0(t, -i\Sigma_n(t)) + \sum_{m=1}^{d} \psi_m(t, -i\Sigma_n(t))X_m(t) \) for
  \( A_n(t) := \sum_{m=n(t)+1}^{n} (T_m - T_{m-1})\alpha_m(t) \),
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  \( \psi_0, \ldots, \psi_d \) exponents corresponding to \( X \)
- Discrete short rate dynamics: is not fully determined by model
Forward process models
– based on affine processes (Eberlein, Özkan 2005, etc.)

- Contin. comp. forward rate: \( f(t, S, T) := \frac{1}{T-S} \log \frac{B(t,S)}{B(t,T)} \)
- Numeraire: discrete money market account

\[
\tilde{S}_0(t) = \exp \left( - \sum_{m=1}^{n} f(T_{m-1}, T_{m-1}, T_m)(T_m - T_{m-1}) \right. \\
- \left. \tilde{f}(t, t, T_n)(T_n - t) \right) \text{ for } T_{n-1} \leq t < T_n
\]

- Forward process dynamics:
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  \]
  \[
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Forward process models
– based on affine processes, ct’d

- Bonds: state variables
- Bond options with time-$T_m$-payoff $H = (B(T_m, T_n) - K)^+$:
  
  $$\pi(t) = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} \pi_{T_m, T_n}(t, z) \frac{K^{1-z}}{z(z-1)} dz$$

  with

  $$\pi_{T_m, T_n}(t, z) = \exp\left(\psi_0(t, T_m, T_n, iz) + \Psi_X(t, T_m, T_n, iz)^\top x(t)\right) \frac{B(t, T_n)^z}{B(t, T_m)^{z-1}}$$

- Swaptions:
  - suppose $X$ univariate (time-inhomogeneous) Lévy,
    $$\sigma_n(t) = \lambda_n \sigma(t)$$
  - then $B(T_m, T_n) = \pi(T_m, T_n, Y(T_m))$ with
    $$Y(t) := \int_0^t \sigma(s) dX(s), \quad \Lambda(t, n) := \lambda_{n(t)+1} + \cdots + \lambda_n,$$
    $$\Sigma_n(t) := \Lambda(t, n) \sigma_n$$ for $T_{n(t)-1} \leq t < T_{n(t)} \leq T_n$,
    $$\pi(T_m, T_n, x) := \frac{B(0, T_n)}{B(0, T_m)} \exp\left(\int_0^{T_m} (A(s, T_n) - A(s, T_m)) \, dt\right) e^{\Lambda(T_m, n)x}$$
  - apply Jamshidian’s representation
Forward process models
– based on affine processes, ct’d

**Bonds: state variables**

Bond options with time-\(T_m\)-payoff \(H = (B(T_m, T_n) - K)^+\):

Value \(\pi(t) = \frac{1}{2\pi i} \int_{\mathbb{R}^+ \cup i\infty} \pi_{T_m, T_n}(t, z) \frac{K^{1-z}}{z(z-1)} \, dz\) with

\[
\pi_{T_m, T_n}(t, z) = \exp\left(\psi_0(t, T_m, T_n, iz) + \psi_X(t, T_m, T_n, iz)^\top x(t)\right) \frac{B(t, T_n)^z}{B(t, T_m)^{z-1}}
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**Swaptions:**

- Suppose \(X\) univariate (time-inhomogeneous) Lévy, \(\sigma_n(t) = \lambda_n \sigma(t)\)
- Then \(B(T_m, T_n) = \pi(T_m, T_n, Y(T_m))\) with
  \(Y(t) := \int_0^t \sigma(s) \, dX(s)\), \(\Lambda(t, n) := \lambda_{n(t)+1} + \cdots + \lambda_n\),
  \(\Sigma_n(t) := \Lambda(t, n) \sigma_n\) for \(T_{n(t)-1} \leq t < T_{n(t)} \leq T_n\),
  \(\pi(T_m, T_n, x) := \frac{B(0, T_n)}{B(0, T_m)} \exp\left(\int_0^{T_m} (A(s, T_n) - A(s, T_m)) \, dt\right) e^{\Lambda(T_m, n)x}\)
- Apply Jamshidian’s representation
Forward process models
– based on affine processes, ct’d

- Bonds: state variables

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  value $\pi(t) = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} \pi_{T_m, T_n}(t, z) \frac{K^{1-z}}{z(z-1)} \, dz$ with

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