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STRESS TESTING THE RESILIENCE OF FINANCIAL NETWORKS

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We propose a simulation-free framework for stress testing the resilience of a financial network to external shocks affecting balance sheets. Whereas previous studies of contagion effects in financial networks have relied on large scale simulations, our approach uses a simple analytical criterion for resilience to contagion, based on an asymptotic analysis of default cascades in heterogeneous networks. In particular, our methodology does not require to observe the whole network but focuses on the characteristics of the network which contribute to its resilience. Applying this framework to a sample network, we observe that the size of the default cascade generated by a macroeconomic shock across balance sheets may exhibit a sharp transition when the magnitude of the shock reaches a certain threshold: Beyond this threshold, contagion spreads to a large fraction of the financial system. An upper bound is given for the threshold in terms of the characteristics of the network.

Keywords: Systemic risk; random graphs; stress test; default risk; macro-prudential regulation.

1. Introduction

In the Supervisory Capital Assessment Program, implemented by the Board of Governors of the Federal Reserve System in 2009 [21], the 19 largest US banks were asked to project their losses and resources under various macroeconomic shock scenarios. The program determined which of the large banks needed to augment its capital base in order to withstand the projected losses. Although underlying this

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stress test was the concern that the failure of these large banks might generate contagion in the US financial system, contagion effects were not directly taken into account when designing the stress tests nor in evaluating the magnitude of losses in the stress scenarios.

Various models for default contagion in banking systems have been proposed in the recent literature, in the framework of *network* models. In this approach, a banking system is modeled as a weighted directed graph in which nodes represent the financial institutions and edges represent exposures between institutions [12, 13]. The fundamental default of certain banks propagates to their counterparties as these write down from their capital the exposures to the defaulted banks [1, 11].

The literature contains many simulation-based studies of contagion in banking networks conducted using central bank data — examples include Elsinger *et al.* [13] for Austria, Cont *et al.* [11] for Brazil, Upper [23] for Germany — as well as similar studies on simulated networks [10, 20]. The conclusions regarding the magnitude of contagion differ across studies, as network topology and regulatory limits, differ from one country to another, but the complexity of the models involved prevent simple insights into the influence of different network characteristics on the results. For the Austrian network, the authors find that among the sources of systemic risk, the direct effect of correlation in the external shocks is far more important than direct contagion effects, which are only secondary. In their case contagious defaults occur only in scenarios where a large number of fundamental defaults occur. In the German network, on the contrary, the default of a single bank can wipe out a significant fraction of the system, so contagion risk is by no means secondary [23].

These studies suggest that some networks are intrinsically fragile and the default of a single bank may trigger a large cascade, whereas other networks might be more resilient to contagion. This intuition is supported by theoretical results on the resilience of networks to contagion [1], and the aim of this work is to integrate such theoretical insights into the stress testing framework, thus shedding some light on the results of such stress tests.

We propose in this work a simple framework for stress-testing the resilience to contagion in a financial network under macroeconomic shocks. Instead of relying on computationally intensive simulations, our approach relies on analytical insights obtained from the asymptotic analysis of the magnitude of default contagion in large networks [1]. Based on the asymptotic analysis of [1], we propose a measure of resilience to contagion, which involves the connectivity of nodes and the proportion of "contagious" links in the network and use it to assess the resilience of the network under macroeconomic shocks. In particular, our methodology does not require to observe the whole network but focuses on the characteristics of the network which contribute to its resilience. Applying this framework to a sample network, we observe that the size of the default cascade generated by a macroeconomic shock across balance sheets may exhibit a sharp transition when the magnitude of the shock reaches a certain threshold: beyond this threshold, contagion spreads to a large fraction of the financial system. An upper bound is given for the threshold in terms of the characteristics of the network. As the resilience measure is a decreasing function of each bank's connectivity and fraction of contagious links, it can be used for monitoring/regulating the financial institutions that pose the highest systemic risk.

The paper is organized as follows. Section 2 presents a model for default contagion in a financial network, following [1, 9]. Section 2.1 presents a probabilistic framework for studying asymptotic behavior of cascades in large networks and recalls the main results of Amini *et al.* [1]. Section 2.2 presents some new results on the size of the default cascade generated by a single node. Section 3 presents a stress testing framework for analyzing the resilience of a network to macroeconomic shocks and discusses two examples: a random infinite network (Sec. 3.2) and a scale-free network whose size is comparable to existing banking networks (Sec. 3.3).

2. Network Models of Banking Systems

Interlinkages across balance sheets of financial institutions may be modeled by a weighted directed graph $\mathbf{g} = (\mathbf{v}, \mathbf{e})$ on the vertex set $\mathbf{v} = [1, \ldots, n]$, whose elements represent financial institutions. Table 1 displays a stylized balance sheet of a financial institution: denoting by $e_{i,j}$ the exposure (in monetary units) of institution i to institution j, the interbank assets of i are given by $A_i = \sum_j e_{i,j}$, whereas $L_i = \sum_j e_{j,i}$ represents the interbank liabilities of i. In addition to these interbank assets and liabilities, a bank may hold other assets and liabilities (such as deposits). The net worth of the bank, given by its **capital** c_i, represents its capacity for absorbing losses before it becomes insolvent. We define the ratio γ_i as

$$c_i = \gamma_i A_i.$$

We will refer to γ_i as "capital ratio" although technically it is the ratio of capital to interbank assets and not total assets. An institution is *insolvent* if its net worth is negative or zero, in which case we set $\gamma_i = 0$.

Definition 2.1. A **financial network** $(\mathbf{e}, \gamma)$ is defined by

- a matrix of exposures $\{e_{i,j}\}_{1 \leq i,j \leq n}$,

Table 1. Stylized balance sheet of a bank.

Assets	Liabilities
Interbank assets $A_i = \sum_j e_{i,j}$	Interbank liabilities $L_i = \sum_j e_{j,i}$
	Deposits D_i
Other assets x_i	Net worth $c_i = \gamma_i A_i$

- a set of capital ratios $\{\gamma_i\}_{1\le i\le n}$ with $\gamma_i > 0$.

The number of an institution's creditors is called its *in-degree*

$$d^-(i) = \#\{j \in \mathbf{v}\, s.t.\, e_{j,i} > 0\},$$

while the *out-degree* of a node i is the number of it's debtors

$$d^+(i) = \#\{j \in \mathbf{v}\, s.t.\, e_{i,j} > 0\}.$$

In a network $(\mathbf{e}, \gamma)$, the set of initially insolvent institutions is represented by

$$\mathbb{D}_0(e, \gamma) = \{i \in \mathbf{v} \,|\, \gamma_i = 0\}.$$

If we denote by R_j the recovery rate for the debt of a market participant j, then j's default induces a loss equal to $(1 - R_j)e_{i,j}$ to its counterparty i. If this loss is greater then i's capital, then i defaults. The set of nodes which become insolvent due to their exposures to initial defaults is

$$\mathbb{D}_1(e, \gamma) = \left\{ i \in \mathbf{v} \,|\, \gamma_i A_i < \sum_{j\in\mathbb{D}_0} (1 - R_j)e_{i,j} \right\},$$

and generally $\mathbb{D}_r$ represents the set of nodes defaulting in round r due to exposures to nodes defaulted in rounds $0, \ldots, r-1$.

It is easy to see that the process finishes at most after $n - 1$ time steps, if the network is of size n, and gives the increasing sequence of default sets

$$\mathbb{D}_0(e, \gamma) \subseteq \mathbb{D}_1(e, \gamma) \subseteq \cdots \subseteq \mathbb{D}_{n-1}(e, \gamma).$$

We assume in what follows that the recovery rate is constant over all nodes and equal to R. Over a short time horizon the recovery rate can be approximated by zero, which is the value we will use in the numerical examples in the next section. The final fraction of defaults at the end of the cascade process is a deterministic function of the exposure matrix and sequence of capital ratios, that we denote

$$\alpha_n(\mathbf{e}, \gamma) = \frac{|\mathbb{D}_{n-1}(e, \gamma)|}{n}.$$

2.1. *Asymptotic analysis of default cascades in large networks*

We summarize here the main results of Amini *et al.* [1]. The financial network $(\mathbf{e}, \gamma)$ is embedded in a sequence of financial networks, indexed by their size $(\mathbf{e_n}, \gamma_\mathbf{n})$. The sequences of in and out-degrees in these networks, also indexed by n, are denoted $\mathbf{d_n^+} = \{d_n^+(i)\}_{i=1}^n$ and respectively $\mathbf{d_n^-} = \{d_n^-(i)\}_{i=1}^n$. Their empirical distribution is given by

$$\mu_n(j, k) := \frac{1}{n}\#\{i : d_n^+(i) = j, d_n^-(i) = k\},$$

and the total number of links in the network of size n by

$$m_n := \sum_i d_n^+(i) = \sum_i d_n^-(i).$$

We assume that the degree sequence $\mathbf{d_n^+} = \{d_n^+(i)\}_{i=1}^n$ and $\mathbf{d_n^-} = \{d_n^-(i)\}_{i=1}^n$ are sequences of nonnegative integers satisfying the following assumptions:

Assumption 2.1. There exists a probability distribution μ on $\mathbb{N}^2$ such that:

1. The empirical proportion of nodes of degree (j,k) tends to $\mu(j,k)$:
$$\mu_n(j,k) \to \mu(j,k) \quad \text{as } n \to \infty;$$
2. Finite average degree property:
$$\exists\, \lambda \in (0,\infty), \quad \sum_{j,k} j\mu(j,k) = \sum_{j,k} k\mu(j,k) =: \lambda;$$
3. $\sum_{i=1}^n d_n^+(i) = \sum_{i=1}^n d_n^-(i)$;
4. $\sum_{i=1}^n (d_n^+(i))^2 + (d_n^-(i))^2 = O(n)$.

The sequences of continuous exposures and capital ratios are mapped into discrete sequences representing the default threshold for each node. We denote by $\Sigma_i^{\mathbf{e}}$ the set of permutations of node i's counterparties in a network $\mathbf{e}$. For each node i and permutation $\tau \in \Sigma_i^{\mathbf{e}}$, we define

$$\Theta(i,\tau,\mathbf{e}) := \min\left\{ k \geq 0, \gamma_i \sum_{j=1}^{d^+(i)} e_{i,j} < \sum_{j=1}^{k} (1-R) e_{i,\tau(j)} \right\}, \tag{2.1}$$

which represents the threshold function: conditional on the order τ in which the counterparties of i may default, this function determines how many defaults i's capital buffer can withstand before i defaults.
Let us define

$$p_n(j,k,\theta) := \frac{\#\{(i,\tau)\,|\, 1 \leq i \leq n, \tau \in \Sigma_i^{\mathbf{e_n}}, d_n^+(i) = j, d_n^-(i) = k, \Theta(i,\tau,\mathbf{e_n}) = \theta\}}{n\mu_n(j,k)j!}. \tag{2.2}$$

Assumption 2.2. There exists a function $p : \mathbb{N}^3 \to [0,1]$ such that for all $j,k,\theta \in \mathbb{N}$ $(\theta \leq j)$

$$p_n(j,k,\theta) \overset{n\to\infty}{\to} p(j,k,\theta).$$

Remark 2.1 (Contagious links). We say that a link is "contagious" if it represents an exposure of a node larger than its capital. It is easy to see that $p_n(j,k,1)$ represents the proportion of "contagious" links leaving nodes with degree (j,k). The limit $p(j,k,1)$ also represents the fraction of nodes with degree (j,k) that default when one counterparty defaults.

Definition 2.2 (Random network ensemble). Let $\mathbb{G}_n(\mathbf{e_n}, \mathbf{d_n^+}, \mathbf{d_n^-})$ be the set of all weighted directed graphs with degree sequence $\mathbf{d_n^+}, \mathbf{d_n^-}$ such that, for any node i, the set of exposures is given by the non-zero elements of line i in the exposure matrix $\mathbf{e_n}$. On a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, we define $\mathbf{E_n}$ as a random network uniformly distributed on $\mathbb{G}_n(\mathbf{e_n}, \mathbf{d_n^+}, \mathbf{d_n^-})$.

We endow the nodes in $\mathbf{E_n}$ with the capital ratios $\gamma_\mathbf{n}$. Then:

$$\forall i = 1, \ldots, n; \quad \{\mathbf{E_n}(i,j), \quad \mathbf{E_n}(i,j) \neq 0\} = \{\mathbf{e_n}(i,j), \quad \mathbf{e_n}(i,j) \neq 0\} \quad \mathbb{P} - a.s.$$

$$\#\{j \in \mathbf{v}, \mathbf{E_n}(j,i) \neq 0\} = d_n^+(j), \quad \text{and} \quad \#\{j \in \mathbf{v}, \mathbf{E_n}(i,j) \neq 0\} = d_n^-(i).$$

The quantity $\alpha_n(\mathbf{E_n}, \gamma_\mathbf{n})$ represents the size of the cascade generated by the default of nodes in the set $\mathbb{D}_0(\mathbf{E_n}, \gamma_\mathbf{n}) = \{i, \gamma_n(i) = 0\}$. The following theorems give the asymptotic behavior of this quantity.

Theorem 2.1. *Define the function*

$$I(\pi) := \sum_{j,k} \frac{k\mu(j,k)}{\lambda} \sum_{\theta=0}^{j} p(j,k,\theta)\mathbb{P}(\mathrm{Bin}(j,\pi) \geq \theta), \tag{2.3}$$

where $\mathrm{Bin}(j,\pi)$ *denotes a binomial variable with parameters* j *and* π.

Consider a sequence of exposure matrices and capital ratios $\{(\mathbf{e_n})_{\mathbf{n}\geq\mathbf{1}}, (\gamma_\mathbf{n})_{\mathbf{n}\geq\mathbf{1}}\}$ *satisfying Assumptions* 2.1 *and* 2.2 *and the corresponding sequence of random matrices* $(\mathbf{E_n})_{n\geq 1}$ *defined on* $(\Omega, \mathcal{A}, \mathbb{P})$ *as in Definition* 2.2. *Let* π^* *be the smallest fixed point of* I *in* $[0,1]$, *i.e.*

$$\pi^* = \inf\{\pi \in [0,1] \,|\, I(\pi) = \pi\}.$$

1. *If* $\pi^* = 1$, *i.e. if* $I(\pi) > \pi$ *for all* $\pi \in [0,1)$, *then asymptotically all nodes default during the cascades*:

$$\alpha_n(\mathbf{E_n}, \gamma_\mathbf{n}) \xrightarrow{p} 1.$$

2. *If* $\pi^* < 1$ *and furthermore* π^* *is a stable fixed point of* I $(I'(\pi^*) < 1)$, *then the asymptotic fraction of defaults satisfies*:

$$\alpha_n(\mathbf{E_n}, \gamma_\mathbf{n}) \xrightarrow{p} \sum_{j,k} \mu(j,k) \sum_{\theta=0}^{j} p(j,k,\theta)\mathbb{P}(\mathrm{Bin}(j,\pi^*) \geq \theta).$$

Definition 2.3 (Resilience measure). We define as the resilience measure the following function of the network's features, which takes values in $(-\infty, 1]$:

$$1 - \sum_{j,k} \frac{jk}{\lambda}\mu(j,k)p(j,k,1).$$

Theorem 2.2. *Under the assumptions of Theorem* 2.1:

- *If the resilience measure is positive, i.e.*

$$1 - \sum_{j,k} \frac{jk}{\lambda}\mu(j,k)p(j,k,1) > 0, \tag{2.4}$$

then for every $\epsilon > 0$, *there exists* N_ϵ *and* ρ_ϵ *such that if the initial fraction of defaults is smaller than* ρ_ϵ, *then* $\mathbb{P}(\alpha_n(\mathbf{E_n}, \gamma_\mathbf{n}) \leq \epsilon) > 1 - \epsilon$ *for all* $n \geq N_\epsilon$.

- *If the resilience measure is negative, i.e.*

$$1 - \sum_{j,k} \frac{jk}{\lambda} \mu(j,k) p(j,k,1) < 0, \tag{2.5}$$

then there exists a connected set C_n of nodes representing a positive fraction of the financial system, i.e. $|C_n|/n \xrightarrow{p} c > 0$ such that, with high probability, any node belonging to this set can trigger the default of all nodes in the set: for any sequence $(\gamma_n)_{n\geq 1}$ such that $\{i, \gamma_n(i) = 0\} \cap C_n \neq \emptyset$,

$$\liminf_n \alpha_n(E_n, \gamma_n) \geq c > 0.$$

Similar results have been obtained using heuristic methods or mean-field approximations in epidemic models on unweighted graphs with arbitrary degree distributions. Gai and Kapadia [14], extending previous work by Watts [24] to the directed graph case, give the following cascade condition:

$$1 - \sum_{j,k} \frac{jk}{\lambda} \mu(j,k) v_j < 0, \tag{2.6}$$

with v_j being the probability that a bank with out-degree j is vulnerable, i.e. exposed to the default of a single neighbor. This condition can be seen as a special case of Condition (2.5) in which the assets and capital buffers are i.i.d. sequences. In such case the convergence Assumption 2.2 is satisfied by the law of large numbers. The results in [14] are derived using generating function methods (see Newman [19]); under the assumption that component sizes are finite, one finds the generating function of the size of a connected component; the point at which the expected size of a connected component diverges marks the phase transition when the giant component appears. The arguments used in [14] are valid for graphs without cycles, i.e. trees. Rigorous proofs of these results and their extension to inhomogeneous weighted directed graphs with prescribed degree sequences are given in [1].

Note that these results are stated in term of the divergence of the **expected** size of a cascade starting from a randomly chosen node, the expectation being computed with respect to the law of a random graph with the given degree distribution. Theorems 2.1 and 2.2 are stronger statements: they show convergence in probability, not just in expectation, of the number of defaults for large networks.

More importantly, we do not assume a specific probabilistic model for the degree sequence or the balance sheet data: actual balance sheet data may be used as an input, under the mild Assumptions 2.1 and 2.2. As such, what appears in other models as the probability of a node to be vulnerable represents here the limit when $n \to \infty$ of the fraction of contagious links, a directly measurable quantity. Moreover as the capital ratios can be prescribed, it suffices to set to zero the capital ratio of certain banks in order to have prescribed defaults. All this is crucial if one wants to use the resilience measure in a stress testing framework.

We note that results closely related to those appearing in Theorems 2.1 and 2.2 have been used in the physics literature to study the size of cascades in networks.

Studies like [6–8] focus on the resilience of scale free networks and specifically the Internet and its robustness to targeted attacks. Meyers *et al.* [18] model the spread of infectious diseases in a semi-directed network, in which the asymmetry of disease transmission is captured by the presence of both undirected and directed edges. Their model generalizes previous work on undirected networks [19] and fully directed networks [22]. The resilience condition in these studies reduces to the existence of a giant component in the percolated graph.

Recently, Gleeson [15] finds the mean size of avalanches on directed unweighted networks in a model in which damage propagates from a fraction ρ of initially damaged nodes and a non-damaged node becomes damaged according to a random threshold depending on its in-degree (we note that damage propagates in [15] in the direction of edges, while in our case in the inverse direction due to our convention to model exposures). The damage propagation functions [15, Eqs. (4) and (5)] are equivalent to the function I in Eq. (2.3) and respectively the limit value for α in Theorem 2.1. The underlying assumption that is made in [15] is that, for large networks with well-behaved degree distributions the state of an input is independent for distinct inputs of a node. In [1] we prove that under the mild Assumptions 2.1 and 2.2 on the degree and exposure sequences, the size of a cascade behaves, in the large limit, as if, at each step, the counterparties of a node default independently of each other. Furthermore, while we prove that the weighted case reduces to the unweighted case, this is not trivial: the function p in Eq. (2.2) is directly computable from balance sheet data and accounts for the heterogeneity of exposures. The model is thus much more general than models with independent degrees and thresholds.

2.2. *Size of default cascade*

We now consider the structure of the skeleton of contagious links. Define the *susceptibility* of a random financial network

$$\chi(\mathbf{E_n}, \gamma_\mathbf{n}) := \frac{1}{n} \sum_{v \in [1,\ldots,n]} |C(v)|, \tag{2.7}$$

with $C(v)$ the default cluster of v containing all nodes from which v is reachable by a directed path of contagious links.

The skeleton of contagious links is the subgraph obtained by retaining only the contagious links in the initial network. Thus, if we consider the new degree sequence for this subgraph, it is still a random graph chosen uniformly from all graphs with this degree sequence [1], so we can still apply asymptotic results for the random configuration model [4, 16]. In particular, Janson [16] shows that the susceptibility of the random graph with given vertex degrees converges under mild conditions to the expected cluster size in the corresponding branching process, which may be defined as a Galton-Watson branching process with initial offspring ξ_0 and general offspring ξ. We define

$$\tilde{\lambda} := \sum_{j,k} j\mu(j,k)p(j,k,1),$$

the average number of contagious links and note that the fraction of contagious links is $T := \frac{\tilde{\lambda}}{\lambda}$. The generating function of the initial offspring ξ_0 is

$$G_0(y) = \sum_{k_0,j,k \geq k_0} \mu(j,k)\binom{k}{k_0}(1-T)^{k-k_0}T^{k_0}y^{k_0} = \sum_{j,k} \mu(j,k)(1-T+Ty)^k,$$

while the generating function of the general offspring is

$$G(y) = \sum_{j,k} \frac{j\mu(j,k)p(j,k,1)}{\tilde{\lambda}}(1-T+Ty)^k.$$

It is easy to see that G_0 represents the generating function of the number of links pointing into a randomly chosen node after bond percolation with probability T (each incoming edge is removed with probability $1-T$ independently of all other incoming edges). In terms of our network model, G represents the generating function of the number of contagious links ending in a node which is start of a randomly chosen contagious link. The probability that such a node has degree (j,k) is given by a weighted version of μ: $\frac{j\mu(j,k)p(j,k,1)}{\tilde{\lambda}}$. We have that

$$\mathbb{E}(\xi) = G'(1) = \sum_{j,k} \frac{j\mu(j,k)p(j,k,1)}{\tilde{\lambda}}kT = \sum_{j,k} \frac{jk\mu(j,k)}{\lambda}p(j,k,1),$$

and

$$\mathbb{E}(\xi_0) = G_0'(1) = \sum_{j,k} k\mu(j,k)T = \tilde{\lambda}$$

For a branching process with initial offspring ξ_0 and general offspring ξ, its susceptibility is given by $1+\frac{\mathbb{E}\xi_0}{(1-\mathbb{E}\xi)_+}$ (see [16, Theorem 3.1], [19]). By virtue of [16, Theorem 3.3] applied to the skeleton of contagious links, under Conditions 2.1 and 2.2, the average cascade size converges in probability (and in fact in L^1, in the subcritical case when $\mathbb{E}(\xi) < 1$) to the susceptibility of the corresponding branching process. We have:

- If the resilience measure is strictly positive,

$$\chi(\mathbf{E_n}, \gamma_\mathbf{n}) \xrightarrow{L^1} \chi_\infty := 1 + \frac{\sum_{j,k} j\mu(j,k)p(j,k,1)}{1-\sum_{j,k}\frac{jk}{\lambda}\mu(j,k)p(j,k,1)}.$$

- If the resilience measure is zero or negative,

$$\chi(\mathbf{E_n}, \gamma_\mathbf{n}) \xrightarrow{p} \infty.$$

We show thus by a different method that the positivity of the resilience measure is a necessary condition for the non-occurrence of global cascades: this condition is equivalent to the non-explosion of the branching process associated to the skeleton of contagious links

$$\mathbb{E}(\xi) < 1.$$

The full distribution of the size of the default cluster can be computed once the generating functions G_0 and G are known (see Bertoin and Sidoravicius [2, Theorem 1] which connects the structure of clusters in random graphs with prescribed degree distributions to branching processes and Newman *et al.* [19] for the derivation in case of branching processes). We define the generating function H of the size of the default cluster generated by a randomly chosen contagious edge, which verifies the condition $H(y) = yG(H(y))$. The generating function H_0 of the size of a default cluster is then given by $H_0(y) = yG_0(H(y))$. If the resilience measure is negative, then the probability of a large scale epidemic triggered by a single node is equal to the explosion probability of the branching process. If we let y^* be the smallest solution of

$$y = \sum_{j,k} \frac{j\mu(j,k)p(j,k,1)}{\tilde{\lambda}}(1 - T + Ty)^k,$$

then the probability of a global cascade is given by

$$1 - \sum_{j,k} \mu(j,k)(1 - T + Ty^*)^k.$$

This last formula confirms the observations in Gleeson [15] that the probability of occurrence of a global cascade strongly depends on the out-degree distribution even when the average cascade size does not, such as in cases where the degree distribution factorizes and the fraction of contagious links does not depend on the out-degrees.

3. Stress Testing

The analytical results presented above may be used to investigate the resilience of a financial network in a stress scenario, without the need for large scale simulation of default cascades. The idea is simply to apply shocks to balance sheets and to compute the impact of these shocks on the resilience measure (Definition (2.3)). Interestingly, it is observed that the final fraction of defaults generated by a fraction ϵ of fundamental defaults undergoes a sharp transition when the size of the shocks exceeds a certain threshold.

We explain how such a stress test may be done and apply the stress test to two example of networks: an infinite random network, and a finite scale-free network whose properties mimic the empirical properties of banking networks [5, 11].

3.1. *Stress testing resilience to macroeconomic shocks*

Consider a banking system in which the ratio $\gamma(i)$ of each bank's capital to its total assets is restricted to be greater than a minimal capital ratio: $\gamma(i) \geq \gamma_{\min}$. If the ratio of institution i's interbank assets to its total assets is denoted by LR_i, then

$$c_i = \gamma_i A_i \frac{1}{LR_i} > 0. \tag{3.1}$$

In a stress testing framework, we consider scenarios in which a given shock is applied to balance sheets of banks, resulting in the loss of a fraction $0 \leq S \leq 1$ of their external assets. To assess how such a stress scenario affects the resilience of the network to contagion, we evaluate the impact on the network of the default of a (small) fraction ϵ of nodes under stress scenarios of variable severity.

Using the notations in Table 1, the remaining capital of bank i is then given by

$$\begin{aligned} c_i(S) &= (A_i + x_i \cdot (1 - S) - L_i) \cdot \epsilon(i) \\ &= \left(A_i + A_i \left(\frac{1}{LR_i} - 1\right) \cdot (1 - S) - \frac{A_i}{LR_i}(1 - \gamma_i)\right) \epsilon(i), \end{aligned}$$

where $\epsilon(i)$ are independent variables with

$$\mathbb{P}(\epsilon(i) = 1) = \epsilon = 1 - \mathbb{P}(\epsilon(i) = 0),$$

where $\epsilon(i) = 1$ indicating thats i is in default in the stress scenario under consideration.

This can be re-written so as to underline the effect of the shock S on the capital

$$c_i(S) = \gamma_i A_i \frac{1}{LR_i} \left(1 - \frac{S}{\gamma_i}(1 - LR_i)\right) \epsilon(i),$$

which means that a loss equal to a fraction S of the external assets translates into a loss equal to a fraction $Z_i := \frac{S}{\gamma_i}(1 - LR_i)$ of the capital buffer. Thus, in the stress scenario characterized by a macroeconomic shock (S, ϵ), the ratio of capital to interbank assets is given by

$$\gamma_i(S, \epsilon) = \gamma_i \left(1 - \frac{S}{\gamma_i}(1 - LR_i)\right) \epsilon(i). \tag{3.2}$$

Starting from this expression, one can use the results of Sec. 2 to evaluate the resilience of the network and the fraction of final defaults as a function of the size of the macroeconomic shock S, without resorting to large scale simulations. In particular, given that the conditions (2.4) and (2.5) will depend on the shock size S, we will see that there is a *threshold* for the magnitude of S above which it destabilizes the network and makes it vulnerable to contagion. This 'phase transition' indicates that a given network has a maximal tolerance for stress; we will see in fact that this threshold may be easily computed from the characteristics of the network.

This approach is applicable to any large network, with an arbitrary distribution of exposures and degrees. To provide some analytical insight into the impact of macroeconomic shocks on the resilience to contagion, we will consider in the next two examples the case where both LR_i and γ_i are constant and equal to LR and γ_{min} respectively. Figures for the lending ratio LR have been given by [14, 17, 23]. We will take $LR = 20\%$ and $\gamma_{min} = 10\%$.

Then the fraction of capital lost in the stress scenario is given by

$$Z = \frac{S}{\gamma_{min}}(1 - LR),$$

so we have

$$\gamma_i(Z) = \gamma_{min}(1 - Z)\epsilon(i).$$

One can observe that in this model, if $Z = 1$, a trivial global cascade ensues, in which all nodes are fundamental defaults: $\forall i, \gamma_i(Z) = 0$. However, as we shall see in the examples in the next sections, a sharp transition in the magnitude of the cascade occur for a threshold value of Z well below 1, which depends on the network characteristics.

3.2. *An example of infinite network*

We first apply the results to an infinite random scale-free network. Such a network may be obtained as the limit when $n \to \infty$ in Blanchard's random graph model [3]. In this model, the limit distribution of the out-degree d^+ is a power law with tail coefficient γ^+

$$\mu^+(j) := P(d^+ = j),$$

while the conditional law of the in-degree is a Poisson distribution

$$P(d^- = k|d^+ = j) = e^{-\lambda(j)}\frac{\lambda(j)^k}{k!},$$

with $\lambda(j) = \frac{j^\alpha \mathbb{E}^{\mu^+}(d^+)}{\mathbb{E}^{\mu^+}((d^+)^\alpha)}$, and α a real parameter. The main theorem in [3] states that the marginal distribution of the in-degree has a Pareto tail with exponent $\gamma^- = \frac{\gamma^+}{\alpha}$, provided $1 \leq \alpha < \gamma^+$. For $\alpha > 0$, one obtains positive correlation between in and out-degrees.

The exposures of each bank with out-degree j are assumed to be independent, and follow a Pareto law. The average exposure is an increasing deterministic function of j. We denote this law F_j.

Note that in this case the limit function $p(j, k, \theta)$ does not depend on the in-degree k (we denote this simply by $p(j, \theta)$), and the function I, whose smallest zero determines the final fraction of defaults (see Theorem 2.1), simplifies to

$$I(\pi) = \sum_j \mu^+(j)\frac{\lambda(j)}{\lambda}\sum_{\theta=0}^{j} p(j, \theta)\beta(j, \pi, \theta)$$

$$= \sum_j \mu^+(j) \frac{j^\alpha}{\mathbb{E}^{\mu^+}((d^+)^\alpha)} \sum_{\theta=0}^{j} p(j,\theta)\beta(j,\pi,\theta)$$

$$= \sum_j \hat{\mu}^+(j) \sum_{\theta=0}^{j} p(j,\theta)\beta(j,\pi,\theta), \tag{3.3}$$

with $\alpha = \gamma^+/\gamma^-$, and $\hat{\mu}^+$ the size-weighted out-degree distribution given by

$$\hat{\mu}^+(j) = \mu^+(j) \frac{j^\alpha}{\mathbb{E}^{\mu^+}((d^+)^\alpha)},$$

which is the probability that the end node of a randomly chosen edge has an out-degree equal to j. Since the out-degree distribution is a Pareto distribution, the size biased out-degree distribution is also Pareto, but with a heavier tail with exponent $\gamma^+ - \alpha$. The resilience condition 2.4 then simplifies to

$$\sum_j \hat{\mu}^+(j) j p(j,1) < 1. \tag{3.4}$$

Under the macroeconomic shock Z, the function $p(j,\theta)$ is given by

$$p(j,\theta) = \mathbb{P}(X(\theta) > \gamma(Z) \sum_{l=1}^{j} X(l) - \sum_{l=1}^{\theta-1} (1-R)X(l) \geq 0),$$

where $(X(l))_{l=1}^{j}$ are i.i.d. random variables with law F_j under $\mathbb{P}$ and $\gamma(Z)$ is given by (3.2). The function $p(j,\theta)$ is plotted in Fig. 1 for a given value of the macroeconomic shock Z. The steep increase with the number of counterparty defaults θ shows how much the system is prone to contagion, especially for the institutions whose assets are concentrated across a small number of counterparties (i.e nodes with small out-degrees).

We consider that a node defaults in the first round with probability ϵ, such that $p(j,0) = \epsilon$, for all j. We plot the function I given by (3.3) for several values of the macroeconomic shock Z in Fig. 2. We notice that the function I has three zeros for smaller values of Z, the smallest being close to zero, and as Z reaches a threshold value Z_c (in this case 42%) its only zero is close to one.

As stated in Theorem 2.1, if the resilience measure is positive, then with high probability, as the initial fraction of defaults tends to 0, no global cascades appear. On the other hand, if the resilience measure is negative, the skeleton of "contagious" links percolates, i.e. represents a positive fraction of the whole system, and we observe global cascades for any arbitrarily small fraction $\epsilon > 0$ of initial defaults chosen uniformly among all nodes. The verification of Theorem 2.1 is shown in Fig. 3. In the non-resilient regime global cascades may occur no matter how small the initial fraction of defaults is. On the contrary, in the resilient regime of the infinite network, if the initial fraction of defaults is small enough, global cascades are not possible. Therefore, the condition of positivity of the resilience measure is a necessary, but not sufficient condition for non occurrence of global cascades.

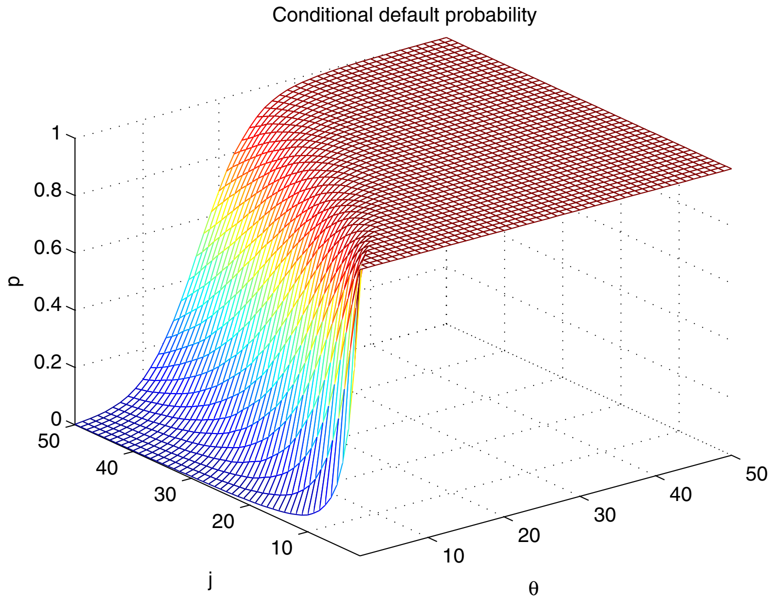


Fig. 1. The conditional probability of default, Minimal capital ratio = 8%, Macroeconomic shock = 20%, Recovery rate = 0.

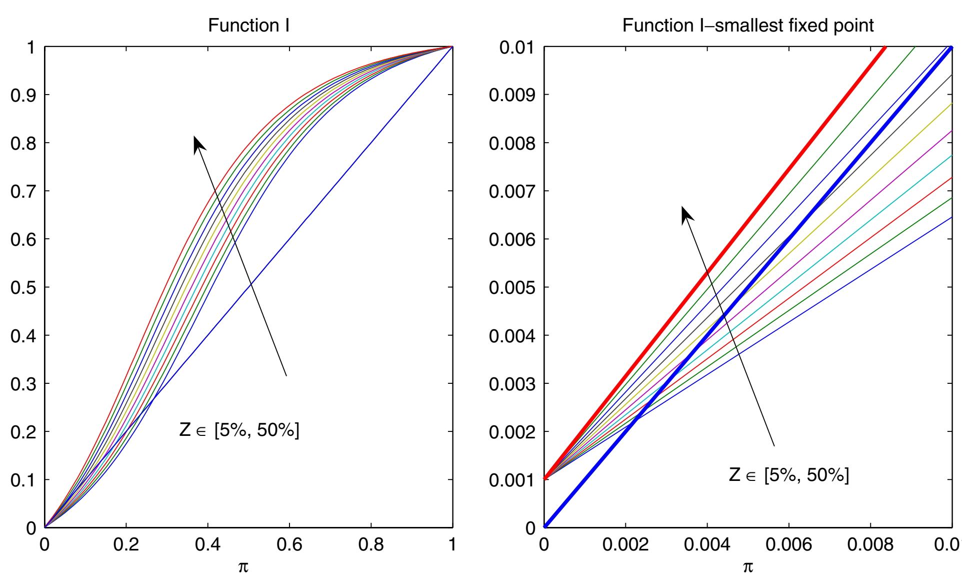


Fig. 2. Function I for varying size of macroeconomic shock. Fraction of initial defaults = 0.1%.

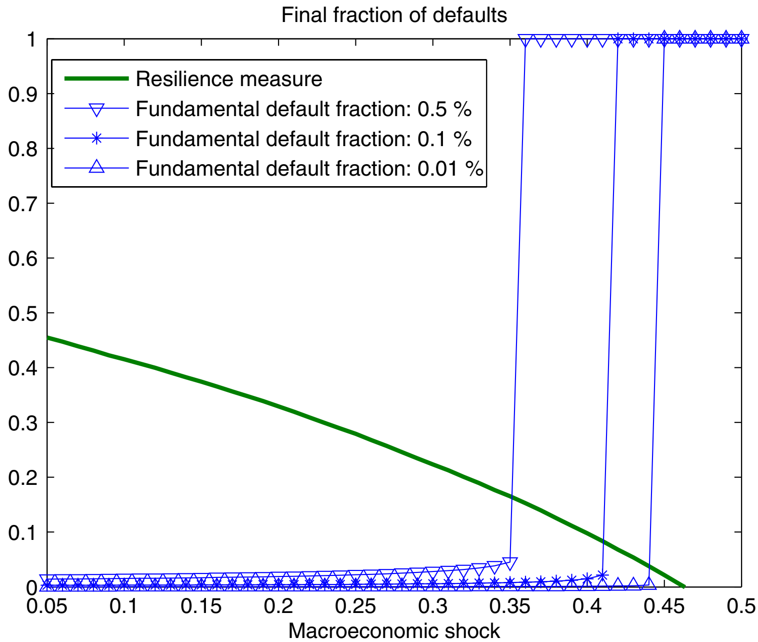


Fig. 3. Final fraction of defaults: infinite network.

3.3. *A finite scale-free network*

We apply the results to a sample scale free network of 2000 nodes with heterogeneous degrees and exposures, generated from Blanchard's random graph model [3]. The empirical distribution of the sample network's degrees and exposures is shown in Fig. 4, and its parameters were based on the analysis of the Brazilian [11] and Austrian [5] networks.

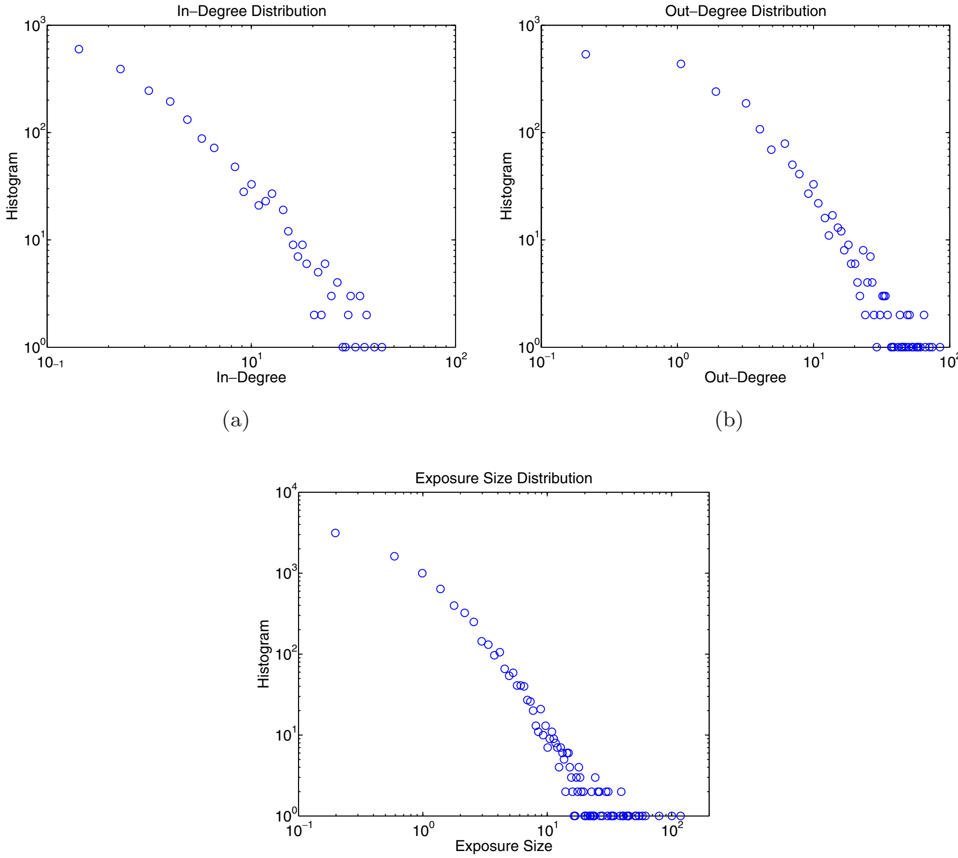
As Fig. 5 shows, we obtain highly correlated asset and liabilities sizes and the average exposure is increasing with the number of debtors for the more connected nodes. These properties are both observed in the empirical data.

In the finite sample, condition 3.4 translates to a condition on the average over all nodes of their number of 'contagious' links with a weight proportional to the out-degree to the power α:

$$\frac{1}{n}\sum_i w_i q_i < 1 \tag{3.5}$$

with $q_i := \#\{j \in v \mid e_{i,j} > c_i\}$ and $w_i := \frac{(d^+(i))^\alpha}{\sum_l (d^+(l))^\alpha}$.

If α is positive, so the more correlated the in-degree and the out-degree are, the more weight is given to the most interconnected nodes. This confirms the intuition that the nodes posing the highest systemic risk are those both overexposed and interconnected, but not necessarily the largest in terms of balance sheet size.



(a) (b) (c)

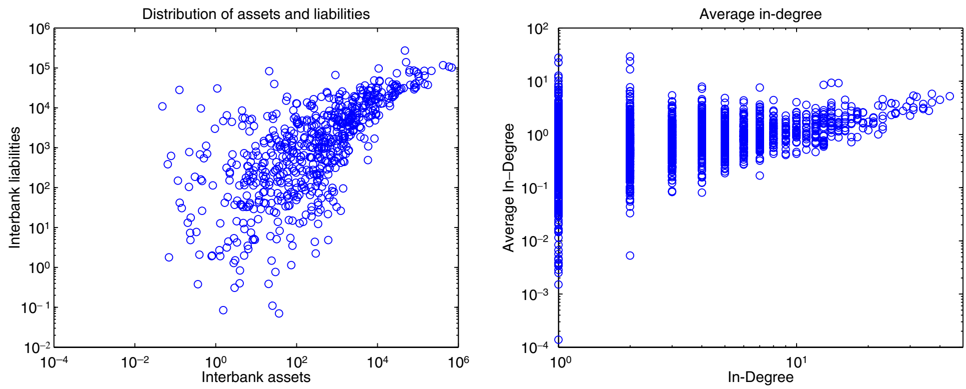
Fig. 4. (a) The distribution of out-degree has a Pareto tail with exponent 3.5, (b) The distribution of the in-degree has a Pareto tail with exponent 2.5, (c) The distribution of the exposures has a Pareto tail with exponent 2.1.

The value $p(j, 1)$ represents the limit fraction of contagious links entering nodes with out-degree j in the limit network. Fig. 6 shows the good accordance between the theoretical values and the values computed in the sample network. This suggests that in practice, there is no need to estimate the parameters of the limit distribution, but instead work directly with the empirical data.

Definition 3.1 (Empirical resilience measure). In a network $(\mathbf{e}, \gamma)$ of size n, we define the empirical resilience measure

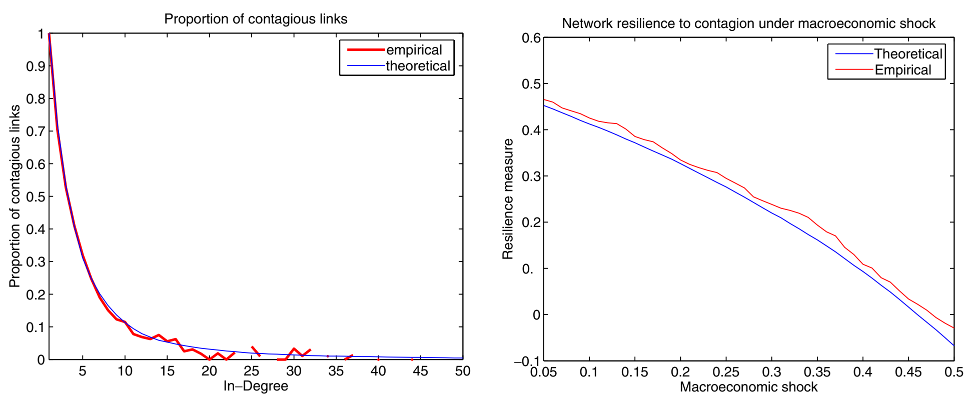
$$1 - \frac{1}{m_n} \sum_i d^-(i) q_i, \tag{3.6}$$

where m_n is the total number of links in the network.



(a) (b)

Fig. 5. (a) Assets and liabilities, (b) Average exposures and connectivity.



(a) (b)

Fig. 6. (a)Proportion of contagious links, (b)Resilience measure for varying size of macroeconomic shock in the sample and limit random network.

We conduct the following simulation on the sample network: two nodes, uniformly selected among all nodes of the network initially default. Then for each value of the macroeconomic shock Z and the corresponding sizes of the capital buffers, we compute the final fraction of defaults. In light of Fig. 3, in the infinite network, for an initial fraction of defaults representing 0.1% of the network, the positivity of the resilience measure is also sufficient for global cascades not to occur.

The results are plotted in Fig. 7 along with the "empirical" resilience measure.

We observe that for the given network and set of initial defaults, there exists a threshold value of the macroeconomic shock, beyond which the contagion spreads to essentially the whole network. By virtue of Theorem 2.1, the threshold value is

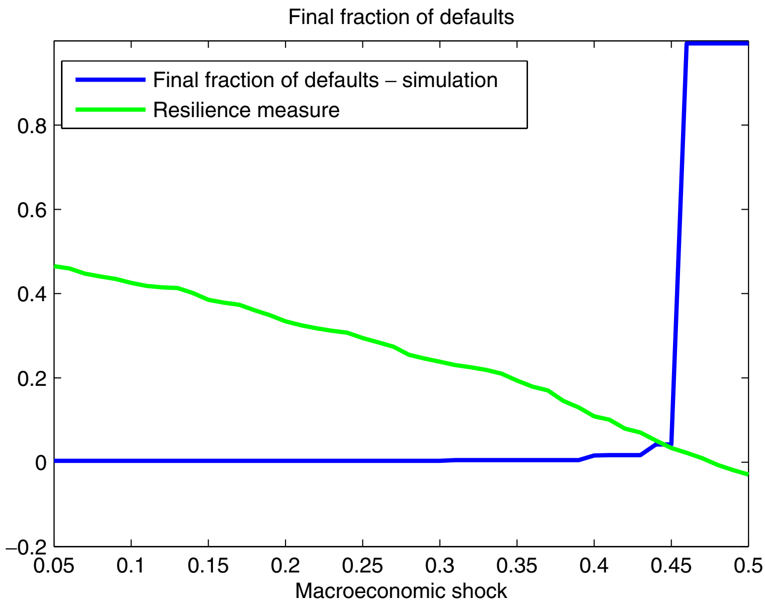


Fig. 7. Final fraction of defaults triggered by an initial fraction of defaults representing 0.1% of the total network.

smaller than the value of Z, for which the empirical resilience measure becomes zero. Also, the proximity of those two values depends strongly on the initial set of defaults. In the considered example, there is a good agreement between the two values, as the initial fraction of defaults is very small.

4. Discussion

We have proposed a framework for evaluating the impact of a macroeconomic shock on the resilience of a banking network to contagion effects. Our approach complements existing stress tests used by regulators [21] and suggests to monitor the capital adequacy of each institution with regard to its *largest exposures*.

In practice, such a stress tests may be implemented in a decentralized fashion by requesting banks to project the effect of a macroeconomic stress scenario on their balance sheets, and report the quantities of interest — mainly the number of exposures exceeding capital in the stress scenario — to the regulator, who can then assess the resilience of the network using our proposed resilience measure. Our criterion for resilience suggests that one need not monitor/know the *entire* network of counterparty exposures, but simply the subgraph of "contagious" links, which is much smaller. This intuition is indeed confirmed by simulation studies on a wide variety of networks [10, 11].

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