Lie II theorem

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Slides available at
http://www.crcg.de/wiki/index.php5?title=User:Zhu

A General phenomenon we have is that when we have a global object, say a group G, it has its corresponding local object G^{loc} a local group, and infinitesimal object $\mathfrak{g}$ a Lie algebra.

$$
\begin{tikzcd}
{} \arrow[r, shift right] \arrow[d] & {} \arrow[l, shift right] \arrow[r, shift right] \arrow[d] & G \arrow[l, shift right] \arrow[d] \\
{} & {} & \text{global}
\end{tikzcd}
$$

∞simal	local	global
Poisson manifold	local symplectic groupoid	symplectic groupoid
Lie algebroid	local Lie groupoid	Lie groupoid
NQ-manifold	local Lie n-groupoid	Lie n-groupoid
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A groupoid consists

- G_0 space of objects $\bullet \quad \bullet$
- G_1 space of arrows $\bullet \longleftarrow \bullet$
- source and target $\mathbf{s}, \mathbf{t} : G_1 \rightrightarrows G_0$
- multiplication $G_1 \times_{G_0} G_1 \xrightarrow{m} G_1$
- identity, inverse
- satisfy usual coherence conditions (e.g. associativity)

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A local Lie groupoid $G^{loc} : G_1^{loc} \rightrightarrows G_0^{loc}$ only has local multiplication:

- $\exists$ an neighborhood V of G_0^{loc} in G_1^{loc} s.t. $V \times_{G_0} V \xrightarrow{m} G_1^{loc}$.

Nerve of a groupoid NG

$$
\begin{array}{c}
X_3 = G_1 \times_{G_0} G_1 \times_{G_0} G_1 \\
\downarrow\downarrow\downarrow\downarrow \quad \uparrow\uparrow\uparrow \\
X_2 = G_1 \times_{G_0} G_1 \\
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X_1 = G_1 \\
\downarrow\downarrow \quad \uparrow \\
X_0 = G_0
\end{array}
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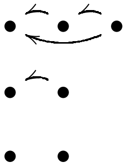

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Simplicial objects

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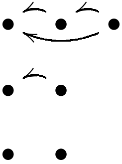

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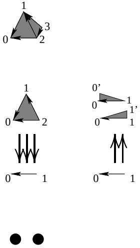

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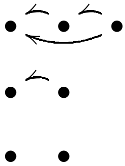

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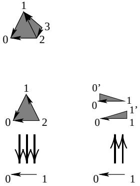

Example: simplical simplex $\Delta[n]$
$\Delta[n]_k :=$ $\{k$-dim faces in an n-dim simplex. $\}$

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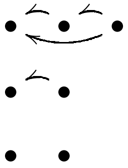

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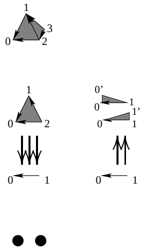

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$\Delta[n]_k :=$ { k-dim faces in an n-dim simplex. }

simplical horn $\Lambda[n,l]$

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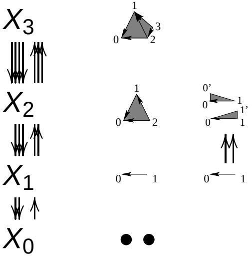
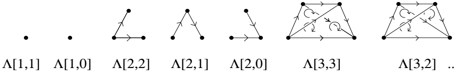
Simplicial objects

Simplicial manifold X





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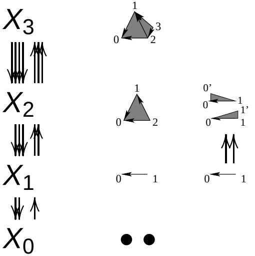

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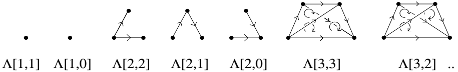

$$\hom(\Delta[n], X) = X_n$$

Simplicial objects

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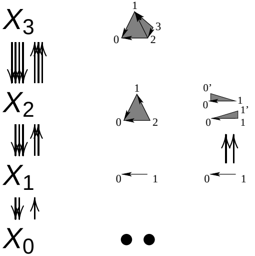

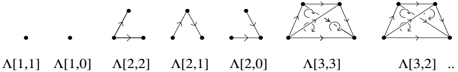

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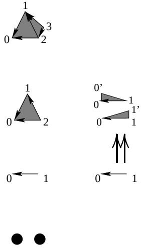

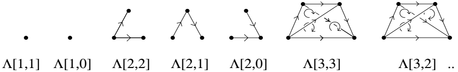

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Definition

Kan condition: horn filling $\exists$ or $\exists!$

Kan(m, j): $\hom(\Delta[m], X) \longrightarrow \hom(\Lambda[m, j], X)$ is a surjective submersion.

Kan$!(m, j)$: $\hom(\Delta[m], X) \longrightarrow \hom(\Lambda[m, j], X)$ is an isomorphism.

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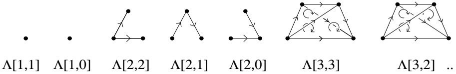

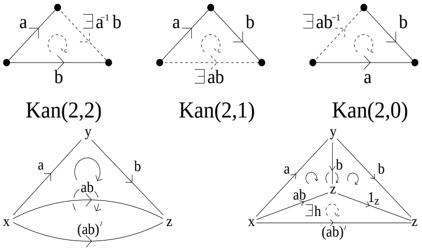

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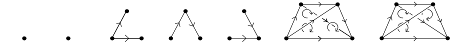

$\Lambda[1,1]$ $\Lambda[1,0]$ $\Lambda[2,2]$ $\Lambda[2,1]$ $\Lambda[2,0]$ $\Lambda[3,3]$ $\Lambda[3,2]$ ..

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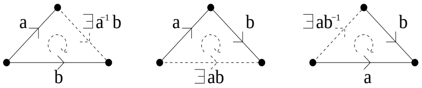
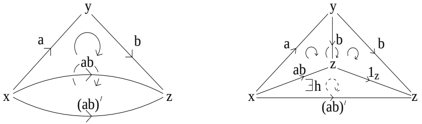


Kan(2,2) Kan(2,1) Kan(2,0)



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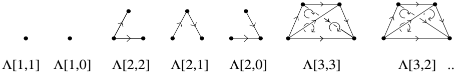
$X = NG \Leftrightarrow X$ satisfies all $Kan(m,j)$ and $Kan!(\geq 2, j)$.

Kan condition



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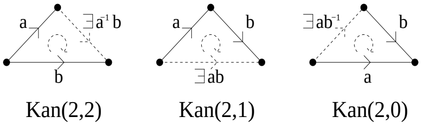
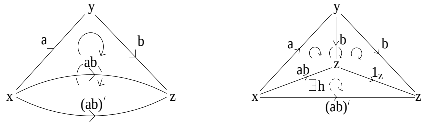


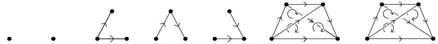


$X = NG \Leftrightarrow X$ satisfies all $Kan(m,j)$ and $Kan!(\geq 2, j)$.

Definition (Duskin, Henriques, Getzler)

Lie n-groupoid: all $Kan(m,j)$ and $Kan!(\geq n+1, j)$.

Kan condition



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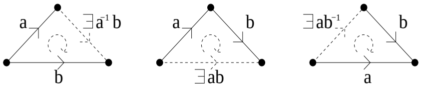

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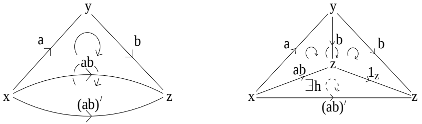

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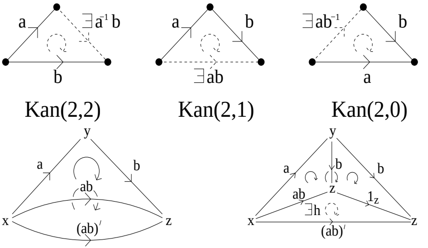
$NG^{loc} \rightsquigarrow NG \rightleftharpoons$ Kanification

Quillen's small object argument



Kanification we fill all the horns

$$
\begin{tikzcd}
\sqcup \Lambda[k,j] \arrow[r] \arrow[d] & X \arrow[d] \\
\sqcup \Delta[k] \arrow[r] & X^1
\end{tikzcd}
.
$$

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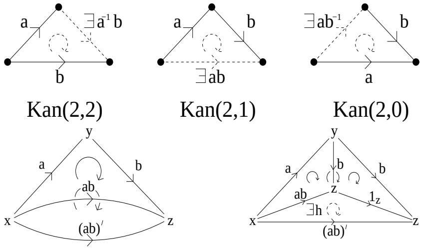
$NG^{loc} \rightsquigarrow NG \rightleftharpoons$ Kanification

Quillen's small object argument



Kanification we fill all the horns

$$\begin{array}{ccc} \sqcup\Lambda[k,j]\times \hom(\Lambda[k,j],X\ \ ) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \sqcup\Delta[k]\times \hom(\Lambda[k,j],X\ \ ) & \to & X^1 \end{array} \quad .$$

$X = NG \Leftrightarrow X$ satisfies all $Kan(m,j)$ and $Kan!(\geq 2, j)$.

Lie n-groupoid satisfies all $Kan(m,j)$ and $Kan!(\geq n+1, j)$.

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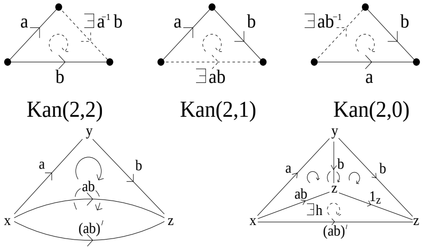

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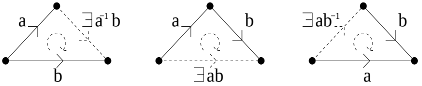
We obtain a sequence of simplicial objects

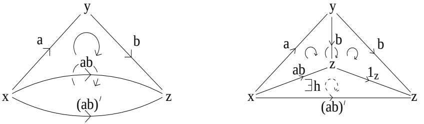
$$X \to X^1 \to X^2 \dots$$

Then $Kan(X) := \operatorname{colim} X^\beta$.

Quillen's small object argument



Kan(2,2) Kan(2,1) Kan(2,0)



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$$\begin{tikzcd}
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\sqcup \Delta[k] \times \hom(\Lambda[k, j], X^\beta) \arrow[r] & X^{1+\beta}.
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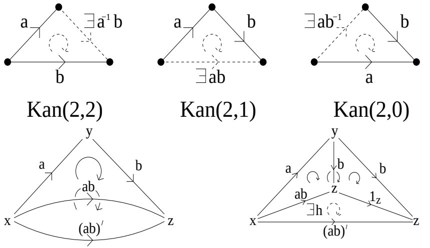

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But $Kan(X)_n$ is not necessarily a manifold!

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We obtain a sequence of simplicial objects

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Then $Kan(X) := \operatorname{colim} X^\beta$.
Can't have $Kan(X)_n$ necessarily a manifold!

Lemma

$Kan(G^{loc})$ is a Kan simplicial manifold.

Lemma

If X is Kan, $Kan(X) \sim_{M.E.} X$ are Morita equivalent.

Lemma

If $X \sim_{M.E.} Y$ are Morita equivalent, then $Kan(X) \sim_{M.E.} Kan(Y)$.

Properties of $Kan(-)$

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Calculation:

$$Kan(X)_0 = X_0$$
$$Kan(X)_1 = X_1 \sqcup X_1 \times_{X_0} X_1 \sqcup (X_1 \times_{X_0} X_1 \sqcup (X_1 \times_{X_0} X_1) \times_{X_0} X_1 \dots) \dots,$$
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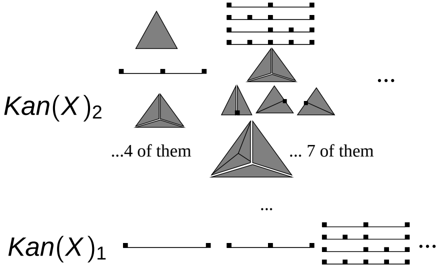

Truncation

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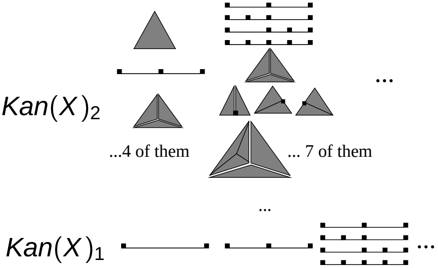

n-Truncation:

$$\tau_n(X)_k = \begin{cases} X_k, & \text{if } k \le n; \\ X_k/X_{k+1}, & \text{if } k > n. \end{cases}$$

τ_n(Kan simplicial manifold) is a n-groupoid!

Truncation

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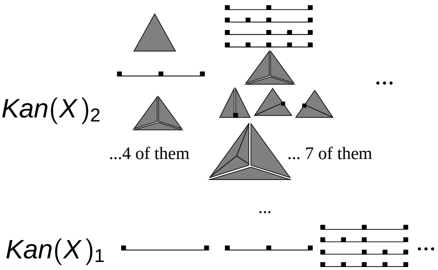

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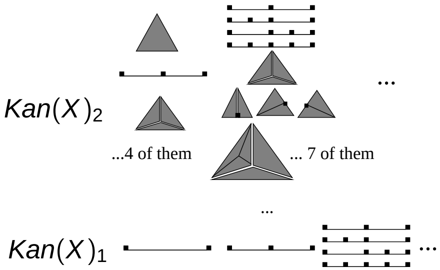

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Problem: manifolds individually are very nice, but they don't form a good category. careful: Quotient, fibre product, limit, ...

$\tau_1(Kan(NG^{loc}))$ in general is only a topological groupoid.

Get Lie stuff

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To recover the local Lie structure, we need

- either, take stacky quotient $\rightsquigarrow$ a stacky Lie groupoid
$$\mathcal{G} \Longrightarrow M \ ,$$
- or, take $\tau_2(Kan(NG^{loc})) \rightsquigarrow$ a Lie 2-groupoid
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They are equivalent under

Theorem (Duskin, [Z1])

Stacky Lie groupoids $\underset{/\sim}{\overset{1-1}{\longleftrightarrow}}$ *Lie 2-groupoids.*

Universal groupoids

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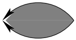
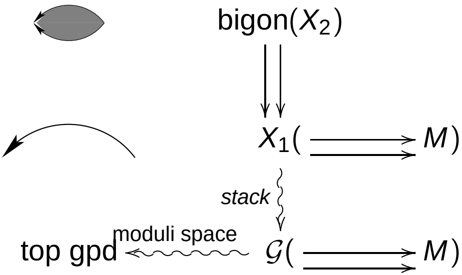


$$\begin{tikzcd} & \mathrm{bigon}(X_2) \arrow[d, shift left] \arrow[d, shift right] \\ & X_1(\rightrightarrows M) \arrow[d, squiggly, "stack"'] \\ \text{top gpd} & \mathcal{G}(\rightrightarrows M) \arrow[l, squiggly, "\text{moduli space}"'] \end{tikzcd}$$

What is $\mathcal{G}$? $\mathcal{G}$ is the universal stacky groupoid of A—Lie algebroid of G^{loc}. $\tau_1(Kan(NG^{loc}))$ is the universal topological groupoid of A (Cattaneo+Felder, Crainic+Fernandes).

simplicial set associated to A



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$$S_n(A) = \mathrm{Hom}_{algd}(T\Delta^n, A),$$

Don't know whether it's a simplicial manifold.

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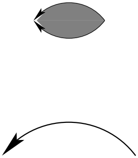

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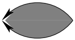

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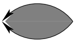

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Different versions of integration

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$Kan(NG^{loc})$ (1) comparing to $S(A)$

1. piecewise finite dimensional,
2. still has strict multiplication: $m : Kan(NG^{loc}) \times_M Kan(NG^{loc}) \to Kan(NG^{loc})$. (True also for its truncations).
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Summary:

1. Universal Lie 2-groupoid of A,

$$G(A) := \tau_2(S(A)) \sim_{M.E.} \tau_2(Kan(NG^{loc}))$$

Equivalent to

2. $\mathcal{G}(A)$ the universal stacky groupoid of A.

Enrichment of

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Lie II

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Theorem (Lie II, [Z2])

A Lie algebroid morphism $A \to B$ and any groupoid G_B of B, give

$$A \to B \rightsquigarrow G(A) \to G_B$$

a morphism of Lie 2-groupoids.

Proof.

$$\begin{tikzcd} A \arrow[r] \arrow[d, phantom, "\Downarrow"] & B \\ G^{loc}(A) \arrow[r] \arrow[rr, bend right] & G^{loc}(B) \arrow[r, hook] & {} \end{tikzcd}$$

$\rightarrow$ $\rightarrow$

$\|$ $\|$

$\square$

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Kan(G^{loc}(A)) \arrow[r] \arrow[d, equal] & Kan(G_B) \arrow[r] & {} \arrow[d, equal] \\
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3. $\tau_1(S(A)) \sim_{M.E.}$ $\tau_1(Kan(NG^{loc}))$ the universal topological groupoid of A.

Theorem (Lie II, [Z2])

A Lie algebroid morphism $A \to B$ and any groupoid G_B of B, give

$$A \to B \rightsquigarrow G(A) \to G_B$$

a morphism of Lie 2-groupoids.

Proof.

$$\begin{tikzcd}
A \arrow[rr] \arrow[drr, phantom, "\Downarrow"] & & B & \\
G^{loc}(A) \arrow[rr] \arrow[rrr, bend right] & & G^{loc}(B) \arrow[r, hook] \arrow[d, phantom, "\Downarrow"] & G_B \\
Kan(G^{loc}(A)) \arrow[rr] \arrow[d, equal] & & Kan(G_B) \arrow[r, "\sim_{M.E.}"] & G_B \arrow[d, equal] \\
{} & & & {}
\end{tikzcd}$$

$\square$

Lie II

Summary:

1. Universal Lie 2-groupoid of A,

$$G(A) := \tau_2(S(A))$$
$$\sim_{M.E.} \tau_2(Kan(NG^{loc}))$$

Equivalent to

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 & & & \Downarrow & \\
\tau_2 Kan(G^{loc}(A)) \arrow[rr] & & \tau_2 Kan(G_B) \arrow[rr, "\sim_{M.E.}"] & & \tau_2 G_B \\
\| & & & & \|
\end{tikzcd}$$

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Proof.

$$\begin{tikzcd}
 & A \arrow[rr] \arrow[rr, phantom, shift right=3, "\Downarrow"] & & B & \\
 & G^{loc}(A) \arrow[rr] \arrow[rrr, bend right=10] & & G^{loc}(B) \arrow[r, hook] \arrow[d, phantom, "\Downarrow"] & G_B \\
\tau_2 Kan(G^{loc}(A)) \arrow[rr] \arrow[d, equal] & & \tau_2 Kan(G_B) \arrow[rr, "\sim_{M.E.}"] & & \tau_2 G_B \arrow[d, equal] \\
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Connectedness

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$$\begin{tikzcd}
& A \arrow[rr] \arrow[rr, phantom, shift right=3ex, "\Downarrow"] & & B & \\
G^{loc}(A) \arrow[rr] \arrow[rrrr, bend right=10] & & G^{loc}(B) \arrow[rr, hook] \arrow[d, phantom, "\Downarrow"] & & G_B \\
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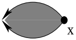
□

Theorem

The universal Lie 2-groupoid $X(A)$ has 2-connected source fibre.

i.e. the Lie groupoid $\mathbf{s}^{-1}(x) = \{$  $\} \Rightarrow \{$  $\}$ has $\pi_0 = \pi_1 = \pi_2 = 1$.

Connectedness

Theorem (Lie II, [Z2])

A Lie algebroid morphism $A \to B$ and any *groupoid G_B of B, give*

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Proof.

$$\begin{tikzcd}
A \arrow[rr] \arrow[rr, phantom, shift right=4, "\Downarrow"] & & B \\
G^{loc}(A) \arrow[r] \arrow[rr, bend right=10] & G^{loc}(B) \arrow[r, hook] \arrow[d, phantom, "\Downarrow"] & G_B \\
\tau_2 Kan(G^{loc}(A)) \arrow[r] \arrow[d, equal] & \tau_2 Kan(G_B) \arrow[r, "\sim M.E."] & \tau_2 G_B \arrow[d, equal] \\
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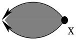
i.e. the Lie groupoid $\mathbf{s}^{-1}(x) = \{$  $\} \Rightarrow \{$  $\}$ has $\pi_0 = \pi_1 = \pi_2 = 1$.

Reason why it's true Let G be the universal Lie group of $\mathfrak{g}$, then

$$\begin{tikzcd}
\text{1conn cover} \\
\tilde{G} \arrow[u, squiggly] \arrow[r, shift left] \arrow[r, phantom, shift right=3, "\Uparrow\text{universal}"] & G \arrow[l, shift left]
\end{tikzcd}$$

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\tau_2 Kan(G^{loc}(A)) \arrow[rr] \arrow[d, equal] & & \tau_2 Kan(G_B^{\sim}) \arrow[rr, "M.E."] & & \tau_2 G_B \arrow[d, equal] \\
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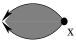
i.e. the Lie groupoid $\mathbf{s}^{-1}(x) = \{$  $\} \Rightarrow \{$  $\}$ has $\pi_0 = \pi_1 = \pi_2 = 1$.

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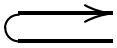
univ. prop. of $G \Rightarrow$  $= id$

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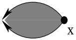
i.e. the Lie groupoid $\mathbf{s}^{-1}(x) = \{$  $\} \Rightarrow \{$  $\}$ has $\pi_0 = \pi_1 = \pi_2 = 1$.

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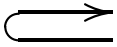
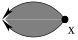
univ. prop. of $G \Rightarrow$  $= id$

univ. prop. of 1-c.c. $\Rightarrow$  $= id$

Connectedness

Theorem

The universal Lie 2-groupoid $X(A)$ has 2-connected source fibre.

i.e. the Lie groupoid $\mathbf{s}^{-1}(x) = \{$  $\} \Rightarrow \{$  $\}$ has $\pi_0 = \pi_1 = \pi_2 = 1$.

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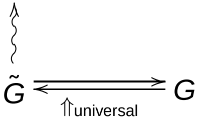

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Now

The source fibre is a 1-groupoid (=differential stack, rather than a 0-groupoid=manifold). We should expect higher covers! E.g. S^2 in this world

$$B\mathbb{Z} \to \tilde{S}^2 \to S^2$$

$\tilde{S}^2 = S^3 \times \mathbb{R} \Rightarrow S^3$ with $\pi_{\leq 2}\tilde{S}^2 = 0$.

:(

Connectedness

Theorem

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1conn cover

$$\tilde{G} \rightleftarrows G$$

$\Uparrow$universal

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:(No homotopy theory of gpds):

Hence can't use it as a proof. The proof is a direct verification.

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