COHOMOLOGY AND REPRESENTATION THEORY

JON F. CARLSON*

0. Introduction

In these lectures we shall consider a few aspects of the interaction between group cohomology and group representation theory. That interaction has grown tremendously in the last thirty years to the point that homological methods are now standard in modular representation theory. The subject is much too large to give a complete picture in the space of a half semester of lectures. Consequently, we will concentrate on the methods and results required for one application: the classification of endotrivial modules. The classification is a statement about modules over group algebras and makes almost no mention of homological algebra or cohomology. Yet its proof relies in fundamental ways on the therory of support varieties, on the computations of the cohomology rings of extraspecial groups and on several items from group cohomology.

An outline of the course is the following. In general we assume a basic knowledge of homological algebra and group representations. The first two or three sections will cover foundational material and be treated mostly as review. In the later sections we encounter some theorems whose proofs, because of time constraints, will be omited or only sketched.

(1) Modular representations of p-groups.
(2) Group cohomology.
(3) Support varieties.
(4) The cohomology ring of a dihedral group
(5) Elementary abelian subgroups in cohomology and representations
(6) Quillen's Dimension Theorem
(7) Properties of support varieties
(8) The rank of the group of endotrivial modules

Throughout these notes, the symbol k denotes a field of prime characteristic p. In general, we assume that k is algebraically closed, though for many of the theorems, this restriction is not necessary. All modules are left unital modules unless stated otherwise. The tensor product $\otimes$ means $\otimes_k$. The k-dual of a kG module or k-vector space M is denoted M^*. All modules will be assumed to be finitely generated. Recall that modules over a finite dimensional algebra satisfy the Krull-Schmidt Theorem. That is, every (finitely generated) module can be written uniquely (up to isomorphism and order of the factors) as a direct sum of indecomposable modules.

Date: April 29, 2005.
This research was supported in part by a grant from NSF.

For most of the basic material, no references are given. The results can be found in one or all of the basic text books on the subject [2, 10, 15].

1. Modules over p-groups

In this section we explore the group algebras of p-groups and their representations. All of the material in this section is standard and can be found in almost any text that deals with modular representation theory. Some of the results of the section hold for all finite groups and not just p-groups. The reader who is unfamiliar with some of the material in this section is encouraged to work through the exercises in some detail.

Throughout this section assume that G is a finite group. We specialize to p-groups later in the section. First we need some basics on group algebras.

Hopf algebras, tensor products and duals. We have a Hopf algebra structure $kG \longrightarrow kG \times kG$ given on basis elements by $g \longrightarrow (g, g)$. This means that if M and N are kG-modules, then so is $M \otimes N$ with the action of $g \in G$ defined by $g(m \times n) = gm \otimes gn$ for $m \in M$ and $n \in N$. Likewise we make $\mathrm{Hom}_k(M, N)$ into a kG-module by letting $(gf)(m) = gf(g^{-1}m)$ for all $f \in \mathrm{Hom}_k(M, N)$ and $m \in M$.

Exercise 1.1. Prove that $\mathrm{Hom}_k(M, N) \cong M^* \otimes N$ by the map which sends $\lambda \otimes n$ to f where $f(m) = \lambda(m)n$ for all $\lambda \in M^*$, $n \in N$ and $m \in M$. Show also that the isomorphism is natural in both variables.

Exercise 1.2. Suppose that $G = \langle x, y\rangle$ is an elementary abelian group of order 4, and k has characteristic 2. Let $M = M_\alpha$ be the kG-module of dimension 2 for which the actions of x and y are given by the matrices

$$x \rightarrow \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad y \rightarrow \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix},$$

for some element $\alpha \in k$. Find a decomposition of $M \otimes M$ into a direct sum of indecomposable modules. Do the same for $M_\alpha \otimes M_\beta$ where α and β are different element of k.

Symmetric and self-injective algebras. Let $\sigma : kG \longrightarrow k$ by $\sigma(\sum a_g \cdot g) = a_1$. That is, σ applied to an element of kG returns the coefficient on the identity element of G. Define a nondegenerate symmetric bilinear form $(\ ,\ ) : kG \times kG \longrightarrow k$ by the rule $(\alpha, \beta) = \sigma(\alpha \cdot \beta)$. It can be seen that the form is G-invariant in the sense that $(\alpha g, \beta) = (\alpha, g\beta)$ for all $g \in G$, $\alpha, \beta \in kG$. The form proves that kG is a symmetric algebra. That is, there is an isomorphism $\phi : kG \cong kG^*$ given by $\phi(\alpha) = (\alpha,\ )$. A consequence of this is the following.

Theorem 1.3. *The group algebra kG is a self-injective algebra. That is, every finitely generated projective module is injective, and conversely, every finitely generated injective module is projective.*

Exercise 1.4. Prove the theorem. Show first that finitely generated free modules are injective using the duality.

Module categories. We let $\mathbf{mod}(kG)$ denote the category of finitely generated kG-modules. Let $\mathbf{stmod}(kG)$ denote the stable category of kG-modules modulo projectives. The objects in $\mathbf{stmod}(kG)$ are the same as those in $\mathbf{mod}(kG)$, but the morphisms from modules M to N are given by

$$\underline{\mathrm{Hom}}_{kG}(M,N) \;=\; \mathrm{Hom}_{kG}(M,N)/\,\mathrm{PHom}_{kG}(M,N)$$

where $\mathrm{PHom}_{kG}(M,N)$ is the set of all homomorphisms from M to N that factor through a projective module.

Induction and Frobenius reciprocity. Suppose that H is a subgroup of G. If M is a kG-module we let M_H denote the restriction of M to a kH-module. If some emphasis is required we use the symbol $M_{\downarrow H}$ to denote the restriction. If N is a kH-module, then the induced module $N^{\uparrow G} = kG \otimes_{kH} N$ is a kG-module with the action of G on the left. Both restriction and induction are functors on the module categories. There are two results relating induction and restriction that are very useful to us. The first is known as Frobenius Reciprocity.

Theorem 1.5. *Let M be a kG-module and N a kH-module. Then*

$$M \otimes N^{\uparrow G} \cong (M_H \otimes N)^{\uparrow G}$$

The isomorphism is given by the map $m \otimes (g \otimes n) \to g \otimes (g^{-1}m \otimes n)$ for all $g \in G$, $m \in M$ and $n \in N$. In the other direction, the map sends $g \otimes (m \otimes n) = gm \otimes (g \otimes n)$.

The other result is known as the Mackey formula.

Theorem 1.6. *Suppose that M is a finitely generated kH-mdoule for H a subgroup of G. Let K be another subgroup of G. Then*

$$(M^{\uparrow G})_{\downarrow K} \cong \sum_{k \times H} ((x \otimes M)_{K \cap xHx^{-1}})^{\uparrow K} = \sum_{KxH} (x \otimes M_{x^{-1}Kx \cap H})^{\uparrow G}$$

where the sum is indexed by the double KxH-cosets in G.

Now notice that if $H = \{1\}$, the identity subgroup and k_H is the trivial kH-module, then $k_H^{\uparrow G} \cong kG$ as left kG-modules. A consequence of this and Frobenius Reciprocity is the following.

Exercise 1.7. Suppose that P is a projective kG-module and that M is any kG-module. Prove that $M \otimes P$ is projective.

Degree shifting. The notation and ideas of this section are vital for the rest of the course. First we recall Schanuel's Lemma.

Proposition 1.8. *Let R be a ring and let M be an R-module. Suppose that P_1 and P_2 are projective modules and that $\theta_1 : P_1 \longrightarrow M$, $\theta_2 : P_2 \longrightarrow M$ are surjective homomorphisms. Let K_i be the kernel of θ_i for $i = 1, 2$. Then $K_1 \oplus P_2 \cong K_2 \oplus P_1$.*

If M is a finitely generated kG-module, then there exists a finitely generated projective cover $\theta : P \longrightarrow M$. That is, P is a projective module of least dimension such that there is a surjective homomophism (theta) onto M. We denote the kernel of θ by $\Omega(M)$. Notice that $\Omega(M)$ has no projective submodule, because if Q were a projective submodule of $\Omega(M)$, then Q would be a projective and also injective submodule of P. Hence Q would be a direct summand of P, thus contradicting the minimality of P. Moreover, $\Omega(M)$ is uniquely defined in the sense that if $\gamma : Q \longrightarrow M$ is any surjective homomorphism with Q projective, then by Schanuel's Lemma the kernel of γ is $\Omega(M) \oplus (\text{proj})$, where by $\oplus(\text{proj})$ we mean the direct sum with some projective module.

Inductively, we define, $\Omega^n(M) = \Omega(\Omega^{n-1}(M))$ for all natural numbers $n > 1$. For the reasons given $\Omega^n(M)$ is well defined up to isomorphism. The module M has an injective hull given by $\theta : M \longrightarrow Q$ where Q a smallest injective (projective) module into which M injects. Then the kernel of θ is denoted $\Omega^{-1}(M)$ and has no injective (hence projective) submodules. Iterating, we define $\Omega^{-n}(M) = \Omega^{-1}(\Omega^{-n+1}(M))$ for $n > 0$. We let $\Omega^0(M)$ be the nonprojective part of M, the direct sum of all of the nonprojective indecomposable summands of M.

With the above definitions and some facts that we know about projective modules, we can prove the following very useful result.

Exercise 1.9. Suppose that M and N are kG-modules and m and n are any integers. Then

(1) $\Omega^m(M) \otimes \Omega^n(N) \cong \Omega^{m+n}(M \otimes N) \oplus (\text{proj})$, and
(2) $(\Omega^n(M))^* \cong \Omega^{-n}(M^*)$.

Definition 1.10. *A kG-module is an endotrivial module provided its k-endomorphism ring is the direct sum of a trivial module and a projective module. That is, M is endotrivial if and only if*

$$\text{Hom}_k(M, M) \cong M^* \otimes M \cong k \oplus (\text{proj})$$

.

The previous exercise shows that for any integer n, $\Omega^n(k)$ is an endotrivial module.

Group algebras of p-groups. Suppose now that G is a p-group. Note that if $x \in G$ then $(x-1)^{p^n} = x^{p^n} - 1 = 0$, if p^n is the order of x. Consequently, the augmentation ideal $I(kG)$ of kG, the ideal generated by all $x - 1$ for x in G, is generated by nilpotent elements. Slightly harder to prove is the following.

Exercise 1.11. Let G be a p-group. Then the augmentation ideal $I(G)$ is a nilpotent ideal of codimension 1 in kG. In particular, $I(G)$ is the radical of kG. It contains every proper ideal of kG, and kG is a local ring.

The exercise implies the following result. The fact that projective modules over a local ring are free modules is well known.

Corollary 1.12. *If G is a p-group, then kG is a local ring, and projective kG-modules are free.*

Another corollary of the exercise is that kG has only one irreducible module, namely, the trivial module k. Another consequence is that the projectivity of a module can be established simply by considering the action of a single element. Let

$\mathfrak{N}_G = \sum_{g \in G} g$ be the sum of all of the elements of G. Note that if $G = \langle g \rangle$ is a cyclic group of order p^n then $\mathfrak{N}_G = (x-1)^{p^n-1}$. As usual, $|G|$ denotes the order of G.

Lemma 1.13. *Let M be a kG-module, then*

$$\operatorname{Dim} \mathfrak{N}_G \cdot M \ \leq \ \frac{1}{|G|} \operatorname{Dim} M.$$

Moreover, we have equality if and only if M is a projective module.

Proof. (Sketch) Note first that $k\mathfrak{N}_G \subseteq kG$ is the unique minimal ideal in kG. In particular it is contained in every other nonzero ideal. Also, the ideal of kG generated by $\mathfrak{N}_G$ has dimension one and is the k-basis for the submodule $\mathfrak{N}_G \cdot M$. Let $a_1, \dots, a_t$ be a basis for $\mathfrak{N}_G \dot{M}$. For each i, let $b_i \in M$ be an element such that $\mathfrak{N}_G b_i = a_i$. Then define $\psi : kG^t \longrightarrow M$ by the formula

$$\psi(\alpha_1, \dots, \alpha_t) \ \longrightarrow \ \sum_{i=1}^{t} \alpha_i b_i.$$

Now we see that ψ is injective because its kernel is zero. Hence, we have an exact sequence

$$0 \longrightarrow kG^t \xrightarrow{\psi} M$$

Because kG^t is injective the sequence splits. Therefore, $M \cong kG^t \oplus M'$ where M' is the cokernel of ψ. The Lemma follows from the fact that $\mathfrak{N}_G \cdot M' = \{0\}$. □

The following is very useful in the later development.

Corollary 1.14. *If $G = \langle x \rangle$ is a cyclic group of order p, then a kG-module M is projective if and only the rank of the matrix of the action of x on M is $((p-1)/p) \operatorname{Dim} M$.*

2. Group cohomology

We assume here that the reader is familiar with basic homological constructions such as chain complexes and chain maps, homology, homotopy of maps, the Künneth formula for tensor product of chain complexes and the functors Ext and Tor. In particular, we often make use of the fact that Ext is the derived functor of Hom as well as being the set of equivalence classes of extensions. Our main concern here is to establish some facts about group cohomology and the relations with representation theory. Most of this relationship is expressed in the notions of support varieties. The notation is as before. In particular, assume that G is a finite group and k is a field of prime characteristic p.

Group cohomology. Suppose that M and N are kG-modules and that

$$(P_*, \varepsilon): \qquad \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

is a projective resolution of M. Recall that for $n > 0$, $\mathrm{Ext}^n_{kG}(M,N)$ is defined to be the (co)homology of the cochain complex $\mathrm{Hom}_{kG}(P_*, N)$. That is,

$$\mathrm{Ext}^n_{kG}(M,N) \;=\; \mathrm{H}^n(\mathrm{Hom}_{kG}(P_*, N)).$$

We also have that

$$\mathrm{Ext}^n_{kG}(M,N) \;=\; \mathrm{H}_n(\mathrm{Hom}_{kG}(M, Q_*)).$$

Where Q_* is an injective resolution of N. Moreover, the Ext functor can be be computed in the stable category as

$$\mathrm{Ext}^n_{kG}(M,N) \;\cong\; \underline{\mathrm{Hom}}_{kG}(\Omega^n(M), N) \;\cong\; \underline{\mathrm{Hom}}_{kG}(M, \Omega^{-n}(N)).$$

The group cohomology of G with coefficients in a module M is defined as

$$\mathrm{H}^n(G,M) \;\cong\; \mathrm{Ext}^n_{kG}(k,M).$$

With the canonical isomorphism $\mathrm{Hom}_{kG}(M,N) \cong \mathrm{Hom}_{kG}(k, M^* \otimes N)$ we can prove that

$$\mathrm{Ext}^n_{kG}(M,N) \;\cong\; \mathrm{H}^n(G, M^* \otimes N).$$

Minimal Resolutions. Let M be a finitely generated kG-module. A projective resolution

$$(P_*, \varepsilon): \qquad \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{\ \varepsilon\ } M \longrightarrow 0$$

of M is a minimal projective resolution if $\partial(P_n) \subseteq \mathrm{Rad}(P_{n-1})$ for every $n > 0$. We construct the minimal projective resolution of M by taking projective covers. That is, let $\varepsilon : P_0 \to M$ be a projective cover of M. Then the kernel of ε is $\Omega(M)$ which has no projective submodules. Now take a projective cover P_1 of $\Omega(M)$ and continue. Some useful properties of minimal resolutions are itemized in the following proposition.

Proposition 2.1. *Let (P_*, ε) be a projective resolution of a finitely generated kG-module M. The following are equivalent statements.*

1. *(P_*, ε) is a minimal projective resolution of M.*
2. *If S is a simple kG-module, then for all $n > 0$*
$$\mathrm{Hom}_{kG}(P_n, S) = \mathrm{Ext}^n_{kG}(M,S).$$
3. *If S is a simple kG-module, then for every $n \geq 0$ the cohomology map*
$$\partial^* : \mathrm{Hom}(P_n, S) \longrightarrow \mathrm{Hom}(P_{n+1}, S)$$
is the zero map.
4. *Let (Q_*, ε') be any projective resolution of M. Then the chain map $\mu : (Q_*, \varepsilon') \longrightarrow (P_*, \varepsilon)$ that lifts the identity map on M is surjective.*
5. *Let (Q_*, ε') be any projective resolution of M. Then any chain map $\nu_* : (P_*, \varepsilon) \longrightarrow (Q_*, \varepsilon')$ that lifts the identity on M is injective.*

Exercise 2.2. Prove the proposition.

The usual example of a minimal resolution is for a cyclic group.

Example 2.3. Suppose that $G = \langle x | x^{p^n} = 1\rangle$ is a cyclic p-group, written multiplicatively. Let $\mathfrak{N}_G = \sum_{g\in G} g \in kG$ be the sum of the elements in G. Notice that $\mathfrak{N}_G = (x-1)^{p^n-1}$. Then we have a periodic projective resolution (X_*, ε) of the form

$$\cdots \xrightarrow{\mathfrak{N}_G} X_3 \xrightarrow{x-1} X_2 \xrightarrow{\mathfrak{N}_G} X_1 \xrightarrow{x-1} X_0 \xrightarrow{\varepsilon} k \longrightarrow 0$$

where for every i, $X_i \cong kG$. That is, the boundary map on X_i for i odd is multiplication by $x-1$, while for i even it is multiplication by $\mathfrak{N}_G$. The exactness of the resolution can be checked from the observation that the elements $\{1, x-1, (x-1)^2, \dots, (x-1)^{p^n-2}, \mathfrak{N}_G\}$ form a k-basis for the free k-module X_i for every i. This resolution may be constructed for any commutative ring of coefficients k.

Exercise 2.4. Let G denote the quaternion group of order 8, given by

$$G \;=\; \langle x, y | x^2 = y^2 = (xy)^2, x^4 = 1\rangle.$$

Note that $yxy^{-1} = x^{-1}$ and that x^2 is the unique element of order 2 in G. Show that the trivial kG-module k has a minimal projective resolution of the form

$$\cdots \longrightarrow KG^2 \longrightarrow kG^2 \longrightarrow kG \longrightarrow kG \longrightarrow kG^2 \longrightarrow kG^2 \longrightarrow kG \longrightarrow k \longrightarrow 0$$

which repeats after every four steps.

Cohomology products. When we think of the cohomology as Ext^*_{kG} in terms of extensions, then it is convenient to think of the products as compositions of sequences (splices). However, there are several other methods of defining the products, some of which are peculiar to group cohomology. Happily, the various definitions are equivalent and each gives us some insight into the multiplicative structure.

With the product structure, $\mathrm{Ext}^*_{kG}(k,k) = H^*(G,k)$ is a graded (associative) ring. We are able to show that the multiplication is graded commutative, in that $\zeta\gamma = (-1)^{\deg(\zeta)\deg(\gamma)}\gamma\zeta$. If M is a kG-module, then $\mathrm{Ext}^*_{kG}(M,M)$ is also a ring and associative. As long as M is finitely generated as a module, its cohomology ring is finitely generated as an algebra. However, it is not always true that $\mathrm{Ext}^*_{kG}(M,M)$ is commutative or graded commutative.

Yoneda splices and compositions of chain maps. Suppose that L, M and N are kG-modules and that $\zeta \in \mathrm{Ext}^n_{kG}(M,L)$, $\gamma \in \mathrm{Ext}^m_{kG}(N,M)$ for $n > 0$, $m > 0$. Then there are exact sequences

$$E : 0 \longrightarrow L \longrightarrow B_{n-1} \longrightarrow \cdots \longrightarrow B_0 \xrightarrow{\nu} M \longrightarrow 0$$

and

$$E' : 0 \longrightarrow M \xrightarrow{\mu} C_{m-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow N \longrightarrow 0$$

which represent ζ and γ respectively. The Yoneda splice or Yoneda composite of the two sequences is a sequence $E \circ E'$ of length $n+m$ given as

$$0 \to L \to B_{n-1} \to \cdots \to B_0 \xrightarrow{\mu\nu} C_{m-1} \to \cdots \to C_0 \to N \to 0$$

Then the product of $\zeta\gamma$ is defined to be the class of the extension $E \circ E'$ in $\mathrm{Ext}^m_{kG}(N, L)$.

Exercise 2.5. Prove that the above product is well defined.

Now suppose that (P_*, ε) and (Q_*, ε') are projective resolutions of the kG-modules M and L, respectively. Let $U^n(M, L)$ denote the homotopy classes of chain maps of degree $-n$ from P_* to Q_*, $n > 0$.

Proposition 2.6. *$U^n(M, L) \cong \mathrm{Ext}^n_{kG}(M, L)$.*

Let $f : P_n \longrightarrow L$ be a cocycle representing a cohomology element $\gamma \in \mathrm{Ext}^n_{kG}(M, L)$. Then let the image of γ be the chain map μ of the following commutative diagram:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & P_{n+1} & \xrightarrow{\partial_{n+1}} & P_n & \xrightarrow{\partial_n} & P_{n-1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \mu_1} & & \downarrow{\scriptstyle \mu_0} & \searrow{\scriptstyle f} & & & \\ \cdots & \longrightarrow & Q_1 & \xrightarrow{\partial_1} & Q_0 & \xrightarrow{\varepsilon'} & L & \longrightarrow & 0. \end{array}$$

So in addition to the Yoneda splice of sequences, we also have a well defined product on cohomology

$$\mathrm{Ext}^m_{kG}(M, L) \otimes \mathrm{Ext}^n_{kG}(N, M) \to \mathrm{Ext}^{m+n}_{kG}(N, L)$$

which is given by composing the chain maps.

Proposition 2.7. *The two products defined on cohomology by Yoneda splice and by the composition of chain maps coincide.*

We should note that the same result could have been completed with injective resolutions. That is, the cohomology product defined by the Yoneda splice operation on exact sequences coincides with the operation which we could define by taking compositions of chain maps on injective resolutions of the modules. Another view of cohomology products is the following.

Products in the stable category. Suppose we look at the cohomology as the Hom functor in the stable category. Recall that, for any m and any kG-modules M and N, we have $\mathrm{Ext}^m_{kG}(N, M) \cong \underline{\mathrm{Hom}}_{kG}(\Omega^m(N), M) \cong \underline{\mathrm{Hom}}_{kG}(\Omega^{m+n}(N), \Omega^n(M))$. We can define a product

$$\begin{aligned} \mathrm{Ext}^n_{kG}(M, L) \otimes \mathrm{Ext}^m_{kG}(N, M) &\cong \\ &\underline{\mathrm{Hom}}_{kG}(\Omega^n(M), L) \otimes \underline{\mathrm{Hom}}_{kG}(\Omega^{n+m}(N), \Omega^n(M)) \\ &\longrightarrow \underline{\mathrm{Hom}}_{kG}(\Omega^{n+m}(N), L) \cong \mathrm{Ext}^{n+m}_{kG}(N, L), \end{aligned}$$

where the middle map is composition of homomorphisms in the stable category. We can prove the following.

Proposition 2.8. *The product on cohomology given by the composition of maps in the stable category coincides with the product defined by Yoneda splice of sequences.*

Products by Hopf algebra structure. Let (P_*, ε) and (P'_*, ε') be projective resolutions of M and M' respectively. Then $H_n(P_*) = 0$ unless $n = 0$ in which case $H_0(P_*) = M$, and similarly for P'_*. So we have that $H_n(P_* \otimes P'_*) = 0$ unless $n = 0$, and $H_0((P_* \otimes P'_*)) = M \otimes M'$ by the Künneth Tensor Formula. Suppose the cohomology classes $\zeta \in \mathrm{Ext}^n_{kG}(M, N)$ and $\gamma \in \mathrm{Ext}^m_{kG}(M', N')$ are represented by cocyles $f : P_n \to N$ and $f' : P'_m \to N'$. Then we have a cocycle $f \otimes f' : P_n \otimes P'_m \to N \otimes N'$. Note that $P_n \otimes P'_m$ is a direct summand of $(P \otimes P')_{n+m} = \sum_{i=1}^{n+m} P_i \otimes P'_{n+m-i}$. Hence we can consider $f \otimes f' : (P \otimes P')_{n+m} \to N \otimes N'$ as the cocycle with support $P_n \otimes P'_m$ as above. Then the class of $f \otimes f'$ is the outer product of ζ and γ.

In the situation that $M \cong M' \cong k$, then we have a cohomology product

$$\mathrm{H}^*(G, N) \otimes \mathrm{H}^*(G, N') \longrightarrow \mathrm{H}^*(G, N \otimes N').$$

Suppose that $\zeta \in \mathrm{H}^m(G, N)$ and $\zeta' \in \mathrm{H}^n(G, N')$ are represented by cocycles $f : P_m \longrightarrow N$ and $f' : P_n \longrightarrow N'$, where (P_*, ε) is a projective resolution of k. Then the product $\zeta \otimes \zeta'$ is represented by the cocycle $\mu \circ (f \otimes f')$ where $\mu : P_* \longrightarrow P_* \otimes P_*$ is a diagonal approximation, a chain map that lifts the identity on k. In topology, μ is often called the Alexander-Whitney map. If (P_*, ε) is the bar resolution, then μ can be given a very explicit form.

Another method for defining the outer product is via the tensor product of complexes. That is, let P_* be a projective resolution of M and $\zeta \in \mathrm{Ext}^n_{kG}(M, N)$ be represented by an exact sequence

$$E : 0 \longrightarrow N \longrightarrow B_{n-1} \longrightarrow \cdots \longrightarrow B_0 \xrightarrow{\eta'} M \longrightarrow 0.$$

Then we have a chain map μ as in the diagram:

$$\begin{array}{ccccccccccccc}
\cdots \longrightarrow & P_{n+1} & \longrightarrow & P_n & \longrightarrow & P_{n-1} & \longrightarrow & & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow 0 \\
 & \downarrow{\scriptstyle \mu_{n+1}} & & \downarrow{\scriptstyle \mu_n} & & \downarrow{\scriptstyle \mu_{n-1}} & & & & \downarrow{\scriptstyle \mu_0} & & \| & \\
E: & 0 & \longrightarrow & N & \longrightarrow & B_{n-1} & \longrightarrow & \cdots & \longrightarrow & B_0 & \longrightarrow & M & \longrightarrow 0
\end{array}$$

That is, if we let $\mathcal{B}$ be the complex

$$\mathcal{B} : 0 \longrightarrow N \longrightarrow B_{n-1} \longrightarrow \cdots \longrightarrow B_0 \longrightarrow 0,$$

then we have a chain map $\mu : P_* \longrightarrow \mathcal{B}$ whose induced map on homology is the identity (on M).

Similarly, if P'_* is a resolution of M', $\gamma \in \mathrm{Ext}^m_{kG}(M', N')$ is represented by a sequence

$$E' : 0 \longrightarrow N' \longrightarrow C_{m-1} \longrightarrow \cdots \longrightarrow C_0 \xrightarrow{\eta'} M' \longrightarrow 0$$

and we let $\mathcal{C}$ be the complex

$$\mathcal{C} : 0 \longrightarrow N' \longrightarrow C_{m-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow 0,$$

then $H_0(\mathcal{C}) = M'$ and there is a chain map $\nu : P'_* \to \mathcal{C}$ that induces the identity on homology. Then $(\mathcal{B} \otimes \mathcal{C})_*$ is a complex with the property that $H_n(\mathcal{B} \otimes \mathcal{C}) = 0$ if

$n \neq 0$ and $H_0(\mathcal{B} \otimes \mathcal{C}) = M \otimes M'$. Therefore we have an exact sequence

$$U : 0 \longrightarrow N \otimes N' \longrightarrow (B_{n-1} \otimes N') \oplus (N \otimes C_{m-1}) \longrightarrow \cdots$$

$$\longrightarrow B_0 \otimes C_0 \xrightarrow{\eta \otimes \eta'} M \otimes M' \longrightarrow 0.$$

That is, the sequence is the augmented complex of $\mathcal{B} \otimes \mathcal{C}$.

Proposition 2.9. *The sequence U represents the cohomology class $\zeta \otimes \gamma$ in the cohomology group $\mathrm{Ext}^{m+n}_{kG}(M \otimes M', N \otimes N')$.*

Exercise 2.10. Verify that all of the above products are equivalent. In other words, fill in the details in the above discussion.

Commutativity of products. Suppose that (P_*, ε) is a projective resolution of the trivial module k.

Proposition 2.11. *The map $\mu : (P \otimes P)_* \longrightarrow (P \otimes P)_*$ given by $\mu(x \otimes y) = (-1)^{\deg(x)\deg(y)} y \otimes x$ is a chain map that lifts the identity on k.*

We can use this to prove the graded-commutativity in the ring $\mathrm{H}^*(G, k)$.

Theorem 2.12. *Suppose that $\zeta \in H^n(G, k)$ and $\gamma \in H^m(G, k)$. Then $\zeta\gamma = (-1)^{mn}\gamma\zeta$.*

Cyclic group cohomology. Now we consider the special case of a cyclic group.

Proposition 2.13. *Let $G = \langle x | x^{p^n} = 1 \rangle$ be a cyclic group. Then*

$$\mathrm{H}^*(G, k) \cong \begin{cases} k[\zeta] & \text{if } p^n = 2 \qquad (\deg \zeta = 1) \\ k[\eta, \zeta]/(\eta^2) & \text{if } p^n > 2 \qquad (\deg \eta = 1, \deg \zeta = 2). \end{cases}$$

Exercise 2.14. Verify the formula by composing chain maps for the products.

Cohomology rings of elementary abelian groups. Suppose that $G = \langle x_1, \ldots, x_n \rangle$ is an elementary abelian group of order p^n. Then $G \cong \langle x_1 \rangle \times \cdots \times \langle x_n \rangle$ is a direct product of cyclic groups of order p. Let $(X^{(i)}_*, \varepsilon_i)$ be a minimal projective $k\langle x_i \rangle$-resolution of k as before. We can see that $(P_*, \varepsilon) = (X^{(1)}, \varepsilon_1) \otimes \cdots \otimes (X^{(n)}, \varepsilon_n)$ is a minimal projective kG-resolution of k. We also have the product formula on the cohomology. Recall the notation $\Lambda(\eta_1, \eta_2, \ldots, \eta_n)$ for the exterior algebra generated by $\eta_1, \ldots, \eta_n$.

Proposition 2.15. *Let $G = \langle x_1, \ldots, x_n \rangle$ be an elementary abelian group of order p^n and let k be a field of characteristic p. Then*

$$\mathrm{H}^*(G, k) \cong \begin{cases} k[\zeta_1, \ldots, \zeta_n] & \text{if } p = 2 \\ \\ k[\zeta_1, \ldots, \zeta_n] \otimes \Lambda(\eta_1, \ldots, \eta_n) & \text{if } p > 2. \end{cases}$$

If $p = 2$, then each ζ_i occurs in degree 1. If $p > 2$, then each η_i is in degree 1 while each ζ_i is in degree 2. We have relations $\eta_i^2 = 0$ and $\eta_i\eta_j = -\eta_j\eta_i$.

Finite generation. The following is a theorem that will be of great use to us. The proof is far too complicated to be presented in the time alloted.

Theorem 2.16. *(Evens, Venkov) The cohomology ring* $\mathrm{H}^*(G,k)$ *is a finitely generated* k*-algebra. Moreover, if* M *is a finitely generated* kG*-module then* $\mathrm{H}^*(G,M)$ *is a finitely generated module over the ring* $\mathrm{H}^*(G,k)$*.*

3. Support Varieties

Because of the Evens-Venkov Theorem on finite generation of group cohomology we can consider the geometry of the cohomology rings and the geometry of cohomology modules. The theorem tells us that cohomology rings are always quotients of finitely generated free graded commutative algebras by finitely generated ideals. In the case that $p = 2$, the graded commutative rings are actually commutative, and the cohomology ring $\mathrm{H}^*(G,k)$ is a quotient of a polynomial ring. When p is odd, $\mathrm{H}^*(G,k)/\operatorname{Rad}(\mathrm{H}^*(G,k))$ is a polynomial ring because every element in odd degree is in the radical. Note also that $\operatorname{Rad}(\mathrm{H}^*(G,k))$ is in every prime or maximal ideal since it is a nilpotent ideal. Consequently, the collections of maximal ideals for $\mathrm{H}^*(G,k)$ is the same as that for $\mathrm{H}^*(G,k)/\operatorname{Rad}(\mathrm{H}^*(G,k))$.

Examples of cohomology rings As an example, consider the cohomology ring of a dihedral 2-group in the case that the coefficient ring k is a field of characteristic 2. It has the form.

$$\mathrm{H}^*(G,k) \cong k[z,y,x]/(zy).$$

It is the quotient of the polynomial ring $k[z.y.x]$ by the ideal generated by zy, and it looks like the coordinate ring of the union of two planes (the z-x plane and the y-x plane) that intersect in a line. The cohomology generators z, y, x are in degrees 1, 1, 2.

For the semi-dihedral group of order 16, the cohomology ring has the form

$$\mathrm{H}^*(G,k) = k[z,y,x,w]/(z^3,\ zy,\ zx,\ x^2 + y^2w).$$

This time the generators z, y, x, w are in degrees 1, 1, 3, and 4.

The cohomology ring of the extraspecial group of order 27 and exponent 3 is generated by elements $z, y, x, w, v, u, t, s, r$ in degrees 1, 1, 2, 2, 2, 2, 3, 3, 6, subject to the relations.

$$\begin{gathered} yz,\ xy - uz,\ wz + uz,\ wy + vz,\ vy + uz, \\ wx + uv - tz - sy,\ w^2 + uw + sz,\ vx + uw + sz, \\ vw + uv - sy,\ v^2 + uw,\ ux - uv + sy,\ ty - sz, \\ uvz - tu + sv,\ u^2z - tv - sw,\ tw + tu,\ sx - sv, \\ tvz - suz,\ tuz - st,\ tuy - suz,\ svz - st, \\ tuw + tu^2,\ tu^2y - stv,\ stx - stv \end{gathered}$$

(and some of these may be redundant). Note that the commutativity relations, which are not trivial, are not in the above list.

The fact that the cohomology ring $\mathrm{H}^*(G,k)$ is a finitely generated k-algebra means that it is noetherian. We let $V_G(k)$ denote the maximal ideal spectrum of $\mathrm{H}^*(G,k)$.

For G an elementary abelian p-group of p-rank n, $V_G(k) \cong k^n$. That is, every maximal ideal of $\mathrm{H}^*(G,k)$ is the kernel of a homomorphism $\mathrm{H}^*(G,k) \longrightarrow k$ which is given by evaluating a polynomial in $\mathrm{H}^*(G,k)/\operatorname{Rad}(\mathrm{H}^*(G,k))$ at a point $\alpha = (\alpha_1, \dots, \alpha_n)$ in k^n. This, of course, depends on k being algebraically closed. For the dihedral group given above, $V_G(k)$ is the union of two planes joined along a line. In any case, $V_G(k)$ is homogeneous affine variety.

Definitions and properties. For kG-modules M and N, a consequence of the Evens-Venkov Theorem is that $\mathrm{Ext}^*_{kG}(M,N)$ is a finitely generated module over $\mathrm{H}^*(G,k) \cong \mathrm{Ext}^*_{kG}(k,k)$. For any kG-module M, let $J(M)$ be the annihilator in $\mathrm{H}^*(G,k)$ of the cohomology ring $\mathrm{Ext}^*_{kG}(M,M)$. We can take $J(M)$ to be the annihilator in $\mathrm{H}^*(G,k)$ of the identity element Id_M. Then we let $V_G(M) = V_G(J(M))$ be the closed subset of $V_G(k)$ consisting of all maximal ideals that contain $J(M)$. Some of the properties of support varieties are given in the following.

Theorem 3.1. *Let L, M and N be kG-modules.*

(1) $V_G(M) = \{0\}$ *if and only if M is projective.*
(2) *If $0 \to L \to M \to N \to 0$ is exact then the variety of any one of L, M or N is contained in the union of the varieties of the other two. Moreover, if $V_G(L) \cap V_G(N) = \{0\}$, then the sequence splits.*
(3) $V_G(M \otimes N) = V_G(M) \cap V_G(N)$.
(4) $V_G(\Omega^n(M)) = V_G(M) = V_G(M^*)$ *where $M^* = \mathrm{Hom}_k(M,k)$ is the k-dual of M.*
(5) *If $V_G(M) = V_1 \cup V_2$ where V_1 and V_2 are non-zero closed subsets of $V_G(k)$ and $V_1 \cap V_2 = \{0\}$, then $M \cong M_1 \oplus M_2$ where $V_G(M_1) = V_1$ and $V_G(M_2) = V_2$.*
(6) *A nonprojective module M is periodic (i.e. for some $n > 0$, $\Omega^n(M) \cong \Omega^0(M)$) if and only if its variety $V_G(M)$ is a union of lines through the origin in $V_G(k)$.*

Proof. Recall that $\mathrm{Ext}^*_{kG}(M,M)$ is finitely generated as a module over $\mathrm{H}^*(G,k)$. So if $V_G(M) = \{0\}$, then for n sufficiently large $\mathrm{Ext}^n_{kG}(M,M) = \{0\}$. Now $\mathrm{H}^*(G,k)$ acts on $\mathrm{Ext}^*_{kG}(N,M)$ through its action on $\mathrm{Ext}^*_{kG}(M,M)$. Consequently for any module N, $\mathrm{Ext}^n_{kG}(N,M) = \{0\}$ for n sufficiently large. This implies that M has finite projective dimension. But as kG is a self-injective ring, M must be projective. This proves (1).

Part (2) is a consequence of the long exact sequence on cohomology. Statement (3) is rather complicated. We will try to indicate why it is true later. For (4) we need only notice that $\mathrm{Ext}^*_{kG}(M,M) \cong \mathrm{Ext}^*_{kG}(\Omega^n(M), \Omega^n(M))$, not only as rings but as modules over $\mathrm{H}^*(G,k)$. Moreover, $\mathrm{Ext}^*_{kG}(M^*, M^*)$ is anti-isomorphic to $\mathrm{Ext}^*_{kG}(M,M)$. The proof of Statement (5) is again more complicated than we are prepared to handle. The last statement is a consequence of the following more general development. □

Complexity. We define the complexity of a module M to be the least integer $s \geq 0$ such that

$$\lim_{n\to\infty} (\mathrm{Dim}\, P_n)/n^s \;=\; 0.$$

It can also be defined as the least $s \geq 0$ such that

$$\lim_{n\to\infty} (\mathrm{Dim}\,\mathrm{Ext}^*_{kG}(N, M))/n^s \;=\; 0,$$

for all kG-modules N, or as the least $s \geq 0$ such that

$$\lim_{n\to\infty} (\mathrm{Dim}\,\mathrm{Ext}^*_{kG}(M, M))/n^s \;=\; 0.$$

It follows from the fact that $\mathrm{Ext}^*_{kG}(M, M))$ is a finitely generated module over $\mathrm{H}^*(G, k)/J(M)$, that the complexity of M is the dimension of the variety $V_G(M)$, which is the same as the Krull dimension of $\mathrm{H}^*(G, k)/J(M)$. Now if M is periodic then its variety is a homogeneous subvariety of $V_G(k)$ of dimension one. It can only be a union of lines.

Exercise 3.2. Prove part (2) of the therorem.

Rank varieties. Assume now that G is an elementary abelian p-group. Let $G = \langle x_1, x_2, \ldots, x_n\rangle$ and for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in k^n$, let $u_\alpha = 1 + \sum_{i=1}^n \alpha_i(x_i - 1)$. Note that u_α is a unit of order p in kG, when $\alpha \neq 0$. Associated to a kG-module M we can define a rank variety

$$V^r_G(M) = \big\{\alpha \in k^n \mid M{\downarrow}_{\langle u_\alpha\rangle} \text{ is not a free } \langle u_\alpha\rangle\text{–module } \big\} \cup \{0\}$$

where u_α is given as above and where $M{\downarrow}_{\langle u_\alpha\rangle}$ denotes the restriction of M to the subalgebra $k\langle u_\alpha\rangle$ of kG.

Exercise 3.3. Prove that kG is free as a $k\langle u_\alpha\rangle$-module, provided $\alpha \neq 0$.

As an example, consider the situation in Exercise 1.2. Here let $\beta = (b_1, b_2)$ and $u_\beta = i1 + b_1(x-1) + b_2(y-1)$. Notice that u_β acts on M by the matrix

$$u_\beta \leftrightarrow \begin{pmatrix} 1 & 0 \\ b_1 + b_2\alpha & 1 \end{pmatrix}$$

Now, u_β acts freely on M if and only if its matrix is conjugate to the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

But this happens if and only if the matrix of $u_\alpha - 1$ has rank 1. In turn, this happens if and only if $b_1 + b_2\alpha \neq 0$. Hence the variety

$$V^r_G(M) = \{(b_1, b_2) \in k^2 | b_1 + b_2\alpha = 0\} = \{b(\alpha, 1) | b \in k\}.$$

For another example let $G = \langle z, y, x\rangle$ be an elementary abelian group of order 8. Choose elements $A, B, C \in k$ and define M to be a kG-module of dimension 4 such that z, y, x act by the matrices

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\1&0&1&0\\0&1&0&1\end{pmatrix}, \quad \begin{pmatrix}1&0&0&0\\0&1&0&0\\A&0&1&0\\0&B&0&1\end{pmatrix}, \quad \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&C&1&0\\1&0&0&1\end{pmatrix}.$$

Then the matrix of $u_\alpha = 1 + \alpha_1(z-1) + \alpha_2(y-1) + \alpha_3(x-1)$ is

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\\alpha_1 + A\alpha_2 & C\alpha_3 & 1 & 0\\ \alpha_3 & \alpha_1 + B\alpha_2 & 0 & 1\end{pmatrix},$$

Again u_α acts freely on M if and only if the rank of the matrix of $u_\alpha - 1$ is 2. But this happens if and only if the 2×2 minor in the lower left corner is not zero. Consequently, $V_G^r(M) = \{(\alpha_1, \alpha_2, \alpha_3) \in k^3 | (\alpha_1 + A\alpha_2)(\alpha_1 + B\alpha_2) + C\alpha_3^2 = 0\}$.

Exercise 3.4. Prove that $V_G^r(M)$ is a homogeneous variety in the case that G is an elementary abelian p-group.

Then we have the following result for any p.

Theorem 3.5. *Let M be any kG-module. If $p = 2$ then, $V_G^r(M) = V_G(M)$ as subsets of k^n. If $p > 2$ then the map $V_G(M) \longrightarrow V_G^r(M)$ which sends α to $\alpha^p = (\alpha_1^p, \ldots, \alpha_n^p)$ is an inseparable isogeny (both injective and surjective). In particular, for $\alpha \neq 0$, $\alpha^p \in V_G(M)$ ($\alpha \in V_G(M)$ if $p = 2$) if and only if $M{\downarrow}_{\langle u_\alpha \rangle}$ is not a free $k\langle u_\alpha \rangle$-module.*

We should emphasize that if v is a unit in kG such that $v \equiv u_\alpha \bmod(\mathrm{Rad}(kG)^2)$ then $M{\downarrow}_{\langle v \rangle}$ is a free $k\langle v \rangle$-module if and only if $\alpha^p \notin V_G(M)$ ($\alpha \notin V_G(M)$ if $p = 2$). So for example the element $x_1x_2x_3$ fails to act freely on M if and only if $(1, 1, 1, 0, \ldots, 0) \in V_G(M)$.

Restrictions to shifted subgroups The proof of the theorem is rather complicated, but we can give a hint as to why it is true. It really has to do with restrictions of cohomology elements to the cyclic subgroups of kG. We call these shifted cyclic subgroups. With some work it is not difficult to prove the following. The proof is a matter of constructing a chain map from the projective resolution of the trivial module for the subgroup generated by u_α to the projective resolution for the trivial module of G.

Proposition 3.6. *Let α be the nonzero element $\alpha = (\alpha_1, \ldots, \alpha_n) \in k^n$, and let $u_\alpha = 1 + \sum_{i=1}^{n} \alpha_i(x_i - 1)$. Then $U = \langle u_\alpha \rangle$ is a cyclic group of order p and kU is a subalgebra of kG. Hence the cohomology ring of U has the form given below. We use the same notation for the cohomology as in 2.15.*

$$H^*(U, k) = \begin{cases} k[\zeta', \eta']/((\eta')^2) & \text{if } p > 2 \\ k[\zeta'], & \text{if } p = 2. \end{cases}$$

For suitable choices of the generators, the restriction map $\mathrm{res}_{kG,kU} : H^*(G,k) \longrightarrow H^*(U,k)$ *has the values* $\mathrm{res}_{kG,kU}(\eta_i) = \alpha_i\eta'$, $\mathrm{res}_{G,U}(\zeta_i) = \alpha_i^p\zeta'$ *for* $p > 2$*. Finally, for* $p = 2$, $\mathrm{res}_{G,U}(\zeta_i) = f(\alpha)\zeta'$*.*

In particular, we have the following.

Corollary 3.7. *With the notation of the proposition, if* $\zeta = f(\zeta_1, \dots, \zeta_n)$ *is a homogeneous element of degree* $2n$ *in* $\mathrm{H}^*(G,k)$ *(degree* n *if* $p = 2$*), then*

$$\mathrm{res}_{G,U}(\zeta) = f(\alpha_1^p, \dots, \alpha_n^p)(\zeta')^n$$

(or

$$\mathrm{res}_{G,U}(\zeta) = f(\alpha_1, \dots, \alpha_n)(\zeta')^n$$

if $p = 2$*).*

The idea behind the proof of Theorem 3.5 is the following. Suppose that $\alpha \in k^n$ is a point in the rank variety $V_G^r(M)$. Suppose that $\zeta \in J(M)$. Then we must have that

$$\mathrm{res}_{G,\langle u_\alpha\rangle}(\zeta) \cdot \mathrm{Id}_M \; = \; 0.$$

But by definition the restriction of M to $\langle u_\alpha\rangle$ is not a projective module. Because modules over the cyclic group $\langle u_\alpha\rangle$ are periodic and have finitely generated cohomology, the only way that this can happen is if $\mathrm{res}_{G,\langle u_\alpha\rangle}(\zeta) = 0$. Consequently, α (or α^p if $p > 2$) is in $V_G(M)$. This proves one of the inclusions. The other inclusion is considerably more complicated. It was proved by Avrunin and Scott [AS].

Dade's Lemma and restrictions One of the reasons why this all works is contained in a result first proved by Dade [13], but which is a consequence of the equality of the varieties 3.5.

Theorem 3.8. *Suppose that* G *is an elementary abelian* p*-group and assume the above notation. A* kG*-module* M *is projective if and only if its restriction to every* $\langle u_\alpha\rangle$ *is projective.*

More generally, maps on groups always induce maps on the maximal ideal spectra and on support varieties. In particular, suppose that H is a subgroup of G. Then the inclusion $H \longrightarrow G$ induces a restriction map on cohomology rings $\mathrm{res}_{G,H} : \mathrm{H}^*(G,k) \longrightarrow \mathrm{H}^*(H,k)$, and a corresponding map $\mathrm{res}^*_{G,H} : V_H(k) \longrightarrow V_G(k)$ on maximal ideal spectra. That is, the pullback of any maximal ideal in $\mathrm{H}^*(H,k)$ is a maximal ideal in $\mathrm{H}^*(G,k)$. Likewise, for M any kG-module, we have a map on the support varieties $V_H(M) \longrightarrow V_G(M)$. A special case is the following, which can be seen very clearly for rank varieties.

Proposition 3.9. *Suppose that* H *is a subgroup of* G *and that* M *is a* kG*-module. Then* $V_H(M) = (\mathrm{res}^*_{G,H})^{-1}V_G(M)$*. In particular,* M_H *is a free* kH*-module if and only if* $\mathrm{res}_{H,G}(V_H(k)) \cap V_G(M) = \{0\}$*.*

We should mention one further thing about modules over elementary abelian subgroups. The result below also extends to module over any finite group. But we need Quillen's Theorem in order to make that extension.

Proposition 3.10. *Suppose that G is an elementary abelian group of order p^n and M is a kG-module. Let $d = \mathrm{Dim}\, V_G(M)$. Then the dimension of M is divisible by p^{n-d}.*

Proof. By Bezout's Theorem there is a linear subspace $V \subseteq k^n = V_G(k)$, such that $V \cap V_G(M) = \{0\}$ and $\mathrm{Dim}(V) = r = n - d$. Let $\alpha(1), \ldots, \alpha(r)$ be a basis for V. For each i let $u_i = u_{\alpha(i)}$. Then $H = \langle u_1, \ldots, u_r \rangle$ is a shifted subgroup with the property that $\mathrm{res}_{G,H}(V_H(k)) \cap V_G(M) = \{0\}$. So by Proposition 3.9, M is free as a kH-module. So the dimension of M is divisible by $|H|$. □

Why quaternion groups have exotic endotrivial modules. The following example is somewhat premature in our course, because a full proof requires the fact that a kG-module M is endotrivial if and only if its restriction to every elementary abelian p-subgroup is endotrivial. This fact will be proved in the next section. Assuming that fact, the example is a very good illustration of what we want to do with the support variety technology.

Suppose that $G = \langle x, y | x^2 = y^2 = (xy)^2, x^4 = 1 \rangle$ is a quaternion group of order 8. Let $z = x^2$ be the central element of order 2 in G. Let $\overline{G} = G/\langle z \rangle$. Then $\overline{G}$ is a fours group, an elementary abelian group of order 4. So we know that for any subgroup of order 4 such as $C = \langle x \rangle$ in G, $\Omega^2(k)_{\downarrow C} \cong k \oplus (kC)^2$, since $\Omega^2(k_C) \cong k$. Let $\overline{M} = (z-1)\Omega^2(k)$. Then $(\overline{M})_{\overline{C}}$ is a free $k\overline{C}$-module, where $\overline{C} = C/\langle z \rangle$. The implication is that $V_{\overline{G}}(\overline{M})$ is not all of $V_{\overline{G}}(k) = k^2$. Hence it must have dimension one and must be a union of lines.

The cohomology ring of $\mathrm{H}^*(G, k)$ is well known and it is easy enough to compute that the dimension of $\Omega^2(k)$ is 9 (see Exercise 2.4). This means that the dimension of $\overline{M}$ is 4. Now if the support variety $V_{\overline{G}}(\overline{M})$ is a union of r lines, then by Theorem 3.1, $\overline{M}$ is a direct sum of r modules each having dimension at least 2. Consequently, it must be that the support variety is a union of at most two lines.

Next we notice that the three $\mathbb{F}_2$-rational lines in $V_{\overline{G}}(k)$ correspond to the actual subgroups of order 2 of $\overline{G}$ which act freely on $\overline{M}$. So they are not in the support variety. On the other hand, $\Omega^2(k)$ is certainly defined over $\mathbb{F}_2$, and so its variety is $\mathbb{F}_2$-rational. That is, it is the zero set in k^2 of a polynomial with coefficients in $\mathbb{F}_2$. Hence the only choice is that $V_{\overline{G}}(\overline{M})$ consists of the two lines through the points $(1, \alpha)$ and $(1, \alpha^2)$ where α is a cube root of unity in $\mathbb{F}_4$.

Therefore $\overline{M} \cong N_1 \oplus N_2$ where $V_{\overline{G}}(N_1)$ is the line through $(1, \alpha)$ and $V_{\overline{G}}(N_2)$ is the line through $(1, \alpha^2)$. The module

$$\hat{M} = M/\{m \in M | (z-1)m = 0\}$$

is isomorphic to $\overline{M}$ with the isomorphism give by multiplication by $z-1$. So $\hat{M} \cong L_1 \oplus L_2$ where $(z-1)L_1 = N_1$ and $(z-1)L_2 = N_2$. So M has the form

$$\begin{array}{ccc} L_1 & & L_2 \\ & k & \\ N_1 & & N_2 \end{array}$$

where multiplication by $z-1$ takes the top to the bottom. Now let U be the kernel of the quotient map $M \longrightarrow L_2$. Then U has the form

$$\begin{array}{ccc} L_1 & & \\ & k & \\ N_1 & & N_2 \end{array}$$

Finally, let $W = U/N_2$. The thing to note is that as a module over $Z = \langle z \rangle$, W has the form $W_Z \cong k \oplus kZ^2$, which is an endotrivial module. Because Z is the unique nontrivial elementary abelian subgroup of G, this information is sufficient to prove that W is an endotrival module (see 5.5 in a later section).

We should note that the modules constructed above, were discovered by Dade [14] using direct computations some thirty years ago. These are the only endotrivial modules for any p-group that can not be defined over the prime field $\mathbb{F}_p$.

Extending Splittings The following is another illustration of how the support varieties can be used. Assume that G is a p-group which has a central cyclic subgroup $Z = \langle z \rangle$ of order p. Let $\overline{G} = G/\langle z \rangle$.

Exercise 3.11. Suppose that N is a kG-module and that N has the property that $(z-1)^{p-1}N = \{0\}$. Hence, N is a module over the algebra $R = k\langle z \rangle/((z-1)^{p-1})$. Assume that N is a free R-module. So N_Z is a direct sum of copies of R. Suppose that $V_{\overline{G}}((z-1)^{p-2}N) = V_1 \cup V_2$ where $V_1 \cap V_2 = \{0\}$. There exist L_1 and L_2 such that $(z-1)^{p-2}N = L_1 \oplus L_2$ where $V_{\overline{G}}(L_1) = V_1$ and $V_{\overline{G}}(L_2) = V_2$. Show that there exist submodules N_1 and N_2 such that $N = N_1 \oplus N_2$ and for $i = 1, 2$, we have that $L_i = (z-1)^{p-2}N_i$.

Hint: Note that $V_{\overline{G}}((z-1)^i N/(z-1)^{i+1}N) = V_{\overline{G}}((z-1)^{p-2}N)$. The statement concerning the existence of L_1 and L_2 is a direct consequence of Theorem 3.1. In the same way, we must have that $(z-1)^{p-3}N/(z-1)^{p-2}N \cong L_1' \oplus L_2'$ where $V_{\overline{G}}(L_i') = V_i$. So there is another exact sequence

$$0 \longrightarrow W \longrightarrow (z-1)^{p-3}N \longrightarrow L_2 \longrightarrow 0$$

where W contains $(z-1)^{p-2}N$ and $W/(z-1)^{p-2}N \cong L_1'$. So we have a sequence

$$0 \longrightarrow L_2 \longrightarrow W/L_1 \longrightarrow L_1' \longrightarrow 0$$

where L_2 is the quotient $(z-1)^{p-2}N/L_1$. Because $V_{\overline{G}}(L_2) \cap V_{\overline{G}}(L_1) = \{0\}$ the last sequence splits. Now continue from here.

4. The cohomology ring of a dihedral groups.

In this section, we briefly describe the mod-2 cohomology ring $\mathrm{H}^*(G,k)$ in the case that G is a dihedral 2-group. This example will be used extensively in what follows. Through assume that $G = \langle x, y | x^2 = y^2 = (xy)^4 = 1\rangle$ is a dihedral group of order 8. Let k be a field of characteristic 2. Let $X = x+1$ and $Y = y+1$. Then we can see that $X^2 = Y^2 = 0$. We can also show the following.

Exercise 4.1. Show that $XYXY = (XY)^2 = \mathfrak{N}_G = YXYX = (YX)^2$.

In fact, we can show further that

Lemma 4.2. *The group algebra kG has the form*

$$kG \;\cong\; k\langle X, Y\rangle/(X^2, Y^2, XYXY - YXYX).$$

That is, kG is the quotient of a polynomial ring in noncommuting variables X and Y by the ideal generated by X^2, Y^2 and $XYXY - YXYX$.

The proof is reasonably straightforward, and mainly comes from noting that the classes of the elements $1, X, Y, XY, XYX, YXY$ and $XYXY$ form a k-basis for the quotient. As there are exactly eight of these elements and because we have seen that X and Y satisfy the relations that we gave, the quotient must be isomorphic to the group algebra.

The projective resolution. Then we can form a kG-projective resolution of the trivial kG-module k as follows.

Proposition 4.3. *A minimal projective resolution of k has the form*

$$\cdots \longrightarrow P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} k \longrightarrow 0$$

where for all n, $P_n \cong (kG)^{n+1}$ and the boundary maps are given by

$$\partial_1 = \begin{pmatrix} X \\ Y \end{pmatrix}, \qquad \partial_2 = \begin{pmatrix} X & 0 \\ YXY & XYX \\ 0 & Y \end{pmatrix}, \qquad \partial_3 = \begin{pmatrix} X & 0 & 0 \\ YXY & X & 0 \\ 0 & Y & XYX \\ 0 & 0 & Y \end{pmatrix}$$

$$\partial_4 = \begin{pmatrix} X & 0 & 0 & 0 \\ YXY & X & 0 & 0 \\ 0 & YXY & XYX & 0 \\ 0 & 0 & Y & XYX \\ 0 & 0 & 0 & Y \end{pmatrix},$$

$$\partial_5 = \begin{pmatrix} X & 0 & 0 & 0 & 0 \\ YXY & X & 0 & 0 & 0 \\ 0 & YXY & X & 0 & 0 \\ 0 & 0 & Y & XYX & 0 \\ 0 & 0 & 0 & Y & XYX \\ 0 & 0 & 0 & 0 & Y \end{pmatrix}, \qquad \cdots$$

Note that because the variables do not commute, the matrices should be multiplied on the right. So for example if $(a, b, c) \in P_2 \cong kG^3$ then consider (a, b, c) as a row vector and multiply

$$\begin{aligned} \partial_2(a,\ b,\ c) &= \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} X & 0 \\ YXY & XYX \\ 0 & Y \end{pmatrix} \\ &= (aX + bYXY,\ bXYX + cY) \in P_1. \end{aligned}$$

Cocycles and chain maps. Now we can compute the generators for the cohomology ring $\mathrm{H}^*(G, k)$. We are going to write our cohomology products as compositions of chain maps, so it is necessary to get a chain map for each of the generators. Note that $P_1 \cong (kG)^2$ so there are two cohomology generators in degree one, represented by the cocycles $\eta_1, \eta_2 : P_1 \longrightarrow k$, given by

$$\eta_1(a, b) = \varepsilon(a) \qquad \text{and} \qquad \eta_2(a, b) = \varepsilon(b).$$

Then we lift the cocycle η_1 to a chain map η_{1*}

$$\begin{array}{ccccccccc} \cdots \longrightarrow & P_2 & \xrightarrow{\partial_2} & P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{\varepsilon} & k & \longrightarrow 0 \\ & \downarrow{\scriptstyle \eta_{1,2}} & & \downarrow{\scriptstyle \eta_{1,1}} & \searrow^{\eta_1} & & & & \\ \cdots \longrightarrow & P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{\varepsilon} & k & & & \end{array}$$

It can be seen that the chain map for η_1 has the form

$$\eta_{1,1}(a, b) = a, \qquad \eta_{1,2}(a, b, c) = (a, bYX), \qquad \eta_{1,3}(a, b, c, d) = (a, b, 0)$$

$$\eta_{1,4}(a, b, c, d, e) = (a, b, cYX, 0), \qquad \eta_{1,5}(a, b, c, d, e, f) = (a, b, c, 0, 0), \qquad \cdots$$

It can be checked that this is a chain map by verifying that $\eta_{1,i-1}\partial_i = \partial_{i-1}\eta_{1,i}$ for each i.

Similarly the chain map for η_2 has the form

$$\eta_{2,1}(a, b) = b, \qquad \eta_{2,2}(a, b, c) = (bXY, c), \qquad \eta_{2,3}(a, b, c, d) = (0, c, d)$$

$$\eta_{2,4}(a, b, c, d, e) = (0, cXY, d, e), \qquad \eta_{1,5}(a, b, c, d, e, f) = (0, 0, d, e, f), \qquad \cdots$$

Exercise 4.4. Verify that the chain maps are as stated.

Products. Now for the products notice that $\varepsilon \circ \eta_{1,1} \circ \eta_{1,2} : P_2 \longrightarrow k$ is given by $\varepsilon \circ \eta_{1,1} \circ \eta_{1,2}(a, b, c) = \varepsilon(a)$. Likewise, $\varepsilon \circ \eta_{2,1} \circ \eta_{2,2}(a, b, c) = \varepsilon(c)$ and $\varepsilon \circ \eta_{1,1} \circ \eta_{2,2}(a, b, c) = \varepsilon(bXY) = 0$. From the last calculation we conclude that $\eta_1\eta_2 = 0$. From the first two calculations we deduce that there is a new generator of cohomology in degree

2 represented by the cocycle $\zeta : P_2 \longrightarrow k$ by $\zeta(a,b,c) = \varepsilon(b)$. As before we can calculate the chain map that lifts ζ. It has the form

$$\zeta_2(a,b,c) = b, \qquad \zeta_3(a,b,c,d) = (b,c)$$

$$\zeta_4(a,b,c,d,e) = (b,c,d), \qquad \zeta_5(a,b,c,d,e,f) = (b,c,d,e), \qquad \cdots$$

We should check at least informally, that this is everything. For example in degree 6, the cocycle that takes (a,b,c,d,e,f,g) to $\varepsilon(c)$ is the composition $\eta_{1.1}\eta_{1,2}\zeta_4\zeta_6$ and so this represents the class $\eta_1^2\zeta^2$. In this way see that

$$\begin{array}{lcr}
(a,b,c,d,e,f,g) \mapsto \varepsilon(a) & \text{represents} & \eta_1^6 \\
(a,b,c,d,e,f,g) \mapsto \varepsilon(b) & \text{represents} & \eta_1^4\zeta \\
(a,b,c,d,e,f,g) \mapsto \varepsilon(c) & \text{represents} & \eta_1^2\zeta^2 \\
(a,b,c,d,e,f,g) \mapsto \varepsilon(d) & \text{represents} & \zeta^3 \\
(a,b,c,d,e,f,g) \mapsto \varepsilon(e) & \text{represents} & \eta_2^2\zeta^2 \\
(a,b,c,d,e,f,g) \mapsto \varepsilon(f) & \text{represents} & \eta_2^4\zeta \\
(a,b,c,d,e,f,g) \mapsto \varepsilon(g) & \text{represents} & \eta_2^6
\end{array}$$

So we see that we have the complete cohomology in degree 6. In general we have

Proposition 4.5. *Let $G = D_8$ the dihedral group of order 8. The cohomology ring of G has the structure*

$$\mathrm{H}^*(G,k) \qquad \cong \qquad k[\eta_1,\eta_2,\zeta]/\mathcal{I}$$

where $\mathcal{I} = (\eta_1\eta_2)$ is the ideal generated by $\eta_1\eta_2$.

We have not attempted to give a rigorous proof of the proposition, but it should be clear that a proof could be constructed from out calculations.

5. Elementary abelian subgroups in cohomology and representations

In this section, we investigate the role that the elementary abelian subgroups of a group play in the representation theory and the cohomology of the group. The main result is a theorem of Quillen [21], written in the early 1970's, showed that the elements of the cohomology ring of a group could be detectected up to nilpotence by their restrictions to the elementary abelian subgroups. The first application to module theory appeared in the paper of Chouinard (see [Ch]) where he proved, among other things, that a kG-module is projective if and only its restriction to every elementary abelian p-subgroup is projective. Other application by Alperin and Evens [AE1, AE2], Avrunin [Av] and others [AS, C1] quickly followed.

The actual theorem that we prove is stated in terms of the existence of a module with a filtration. The interpretation as a result about cohomology takes some effort. The proof uses a theorem of Serre on the vanishing of a certain product of Bockstein elements. There are several ways of deriving Serre's Theorem, all of which are too complicated to treat in this short course. The Bockstein elements will be explained in terms of certain exact sequences of induced modules which are easy to understand. The approach to the theory was developed in [C2].

The Main Theorem. First we state our main theorem, and we can see a few examples.

Theorem 5.1. *There exists an integer τ, depending only on G, and a finitely generated kG-module V such that the direct sum $k \oplus V$ has a filtration*

$$\{0\} = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_\tau = k \oplus V$$

with the property that for each $i = 1, \ldots, \tau$, there is an elementary abelian p-subgroup $E_i \subseteq G$ and a kE_i-module W_i such that

$$L_i/L_{i-1} \;\cong\; W_i^{\uparrow G}.$$

The modules $V, W_1, \ldots, W_\tau$ are finitely generated.

So the theorem says that the trivial module k is a direct summand of a module that is filtered by modules induced from elementary abelian p-subgroups. The first thing we should prove is that the same holds for any module.

Corollary 5.2. *Let M be any kG-module. Then M is a direct summand of a module that is filtered by modules induced from elementary abelian p-subgroups. Moreover if M is finite dimensional, then such a filtration can be found where every module in the filtration is of finite dimension.*

Proof. Assume that τ, V, L_i, etc. are exactly as in the statement of the theorem. Now tensor everything with M. We get that $M \otimes (k \oplus V)$ has a filtration

$$\{0\} = M \otimes L_0 \subseteq M \otimes L_1 \subseteq \cdots \subseteq M \otimes L_\tau = M \otimes (k \oplus V).$$

Now it is a matter of verifying that

$$M \otimes L_i/(M \otimes L_{i-1}) \;\cong\; M \otimes W_i^{\uparrow G} \;\cong (M \otimes W_i)^{\uparrow G},$$

using Frobenius Reciprocity. □

Example 5.3. Suppose that $G = \langle x | x^9 = 1\rangle$ and $p = 3$. Then let $X = x - 1$ and observe that $kG \cong k[X]/(X^9)$. Let $V = \Omega(k) \cong kG/(X^8)$. Let a be a generator of k and let b be a generator of V. The $k \oplus V$ has a basis consisting of $a, b, Xb, X62b, X^3b, X^4b, X^5b, X^6b, X^7b$. Let L_1 be the submodule generated by $a + X^5b$. Then L_1 has a basis $a + X^5b, X^6b, X^7b$ and $L_1 cong k_H^{\uparrow G}$ where $H = \langle x^3\rangle$. Likewise, if we let L_2 be the submodule generated by X^3b and a, then $(k \oplus \Omega(k))/L_2$ and L_2/L_1 are both isomorphic to $k_H^{\uparrow G}$. Hence, the theorem holds in this case.

Some consequences of the Main Theorem. Now as an application we can prove the theorem of Chouinard.

Theorem 5.4. *Suppose that M is a kG-module. Then M is projective if and only if M_E is projective for every elementary abelian p-subgroup E of G.*

Proof. If M is a projective kG-module, then its restriction to any subgroup H is a projective kH-module. Consequently, the challenge is to prove the other direction. So assume that M_E is a free kE-module for every elementary abelian p-subgroup E of G. Let τ, V, L_i, etc. be exactly as in the theorem. As in the last corollary, the strategy is to tensor everything with M. This time, for each i we have that

$$M \otimes L_i/(M \otimes L_{i-1}) \;\cong\; M \otimes W_i^{\uparrow G} \;\cong (M_{E_i} \otimes W_i)^{\uparrow G}.$$

Because M_{E_i} is a free kE_i-module, we have that $M \otimes L_i/(M \otimes L_{i-1})$ is a free kG-module. This implies that $M \oplus (M \otimes V)$ is a free module. It follows that M is projective. □

The next is really a corollary of the corollary.

Corollary 5.5. *A kG-module M is an endotrivial module if and only if its restriction to every elementary abelian p-subgroup of G is an endotrivial module.*

Proof. It should be clear from the definition that if M is an endotrivial module, then its restriction to every subgroup of G is an endotrivial module. Therefore we can assume that M_E is an endotrivial module for every elementary abelian p-subgroup of G. In particular, for any elementary abelian subgroup $E \neq \{1\}$, this says that $\mathrm{Hom}_k(M,M)_E \cong (M \otimes M^*)_E \cong k_E \oplus (\mathrm{proj})$, and the dimension of M must be congruent to 1 or -1 modulo p. So, $\mathrm{Dim}\, M$ is not divisible by p. Now, we have a trace map

$$\mathrm{Hom}_k(M,M) \xrightarrow{Tr} k$$

which simply sends any matrix α to its trace. In addition, the identity homomorphism on M is fixed by elements of G. Consequently, the map

$$\gamma : \quad k \longrightarrow \mathrm{Hom}_k(M,M)$$

such that $\gamma(a) = (a/\mathrm{Dim}\, M)\,\mathrm{Id}_M$, is a homomorphism. In addition $Tr(\gamma(a)) = a$ for all $a \in k$. Let K denote the kernel of the trace map. We have an exact sequence

$$0 \longrightarrow K \longrightarrow \mathrm{Hom}_k(M,M) \xrightarrow{Tr} k \longrightarrow 0$$

which is split and is also split on restriction to any elementary abelian p-subgroup of G. Hence, $\mathrm{Hom}_k(M,M)_E \cong k_E \oplus K_E$ It follows that K_E is a projective module for every elementary abelian p-subgroup E of G. By Chouinard's Theorem, K is a projective, and hence also injective kG-module. Therefore the sequence splits and M is an endotrivial module. □

Proof of the Main Theorem: Reduction to p-groups. We have stated the main theorem for an arbitrary finite group, but it is really only necessary to prove it for p-groups. That is we have the following.

Proposition 5.6. *Suppose that Theorem 5.1 is true for the Sylow p-subgroup P of a finite group G. Then the theorem holds for G.*

Proof. By our assumption, there exists a number τ, a kP-module V, elementary abelian subgroups E_i of P and kE_i modules W_i such that $k_P \oplus V$ has a filtration

$$\{0\} = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_\tau = k_P \oplus V,$$

where $L_i/L_{i-1} \cong W_i^{\uparrow P}$. Now we induce the entire structure to G obtaining the sequence

$$\{0\} = L_0^{\uparrow G} \subseteq L_1^{\uparrow G} \subseteq \cdots \subseteq L_\tau^{\uparrow G} = k_P^{\uparrow G} \oplus V^{\uparrow G}.$$

Now the first point is that k is a direct summand of $k_P^{\uparrow G}$. That is we have a map $\phi : k_P^{\uparrow G} \cong kG \otimes_{kP} k \longrightarrow k$ given by $\phi(g \otimes a) = ga = a$ for $g \in G$ and $a \in k$. and a map $\theta : k \longrightarrow k_P^{\uparrow G}$ by $\theta(a) = (1/|G : P|) \sum(g \otimes a)$, where the sum is over a complete set of representatives of the left cosets of P in G. It can be checked that $\phi\theta$ is the identity on k. Next we check that

$$(L_i^{\uparrow G})/(L_{i-1}^{\uparrow G}) \cong (L_i/L_{i-1})^{\uparrow G} \cong (W_i^{\uparrow P})^{\uparrow G} \cong W_i^{\uparrow G}$$

by the transitivity of the induction. Hence the theorem holds for G. □

Proof of the Main Theorem: Serre's Theorem. Serre's Theorem, which we state below, is the result that tells us that the cohomology and representation theory of an elementary abelian p-group is different from that of an arbitrary p-group. It is phrased in terms of the vanishing of a certain product of cohomology elements, but we will see that it has deep implications for the structure of modules over group algebras.

In our development we will not give a proof of Serre' Theorem. The original proof used the action of the Steenrod algebra on the group cohomology. Other proofs have been given by Kroll [Kr] using Chern classes and by Pakianathan and Yalcin [PY] using LS-categories. Here is the statement of the theorem.

Theorem 5.7. [Se1]. *Let $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ be the field with p elements. Suppose that G is a p-group which is not elementary abelian. Then there is a sequence $\gamma_1, \gamma_2, \dots, \gamma_n \in H^1(G, k)$ of nonzero elements such that if $p = 2$ then*

$$\gamma_1 \cdots \gamma_n = 0,$$

while if $p > 2$, then

$$\beta(\gamma_1) \cdots \beta(\gamma_n) = 0,$$

where β is the Bockstein map.

In the case that G is a dihedral group of order 8, we know that $\eta_1\eta_2 = 0$ where η_1 and η_2 are the two degree one elements as in the last section. So Serre's Theorem is true in this case.

Suppose first that $p = 2$ and that $\zeta \in \mathrm{H}^1(G, k)$ where $k = \mathbb{F}_2$. Then ζ as an element in $\mathrm{Ext}^1_{kG}(k, k)$ is represented by an exact sequence of the form in the bottom row of the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega(k) & \longrightarrow & kG & \xrightarrow{\varepsilon} & k & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \zeta'} & & \downarrow{\scriptstyle \theta} & & \downarrow & & \\ 0 & \longrightarrow & k & \longrightarrow & U & \longrightarrow & k & \longrightarrow & 0, \end{array}$$

where ζ' represents ζ and U is the pushout. Now note that U has dimension 2. Hence $(\mathrm{Rad}\, kG)^2$ is in the kernel of θ. Recall also that

$$xy - 1 \equiv (x-1) + (y-1) \qquad \mathrm{mod}(\mathrm{Rad}(kG))^2.$$

So the set $\{x : x \in G \mid \theta(x-1) = 0\}$ is actually a maximal subgroup H of G. Moreover, $U \cong k_H^{\uparrow G}$ is the induced module from H. As a consequence, we see that ζ is represented by the sequence

$$0 \longrightarrow k \longrightarrow k_H^{\uparrow G} \longrightarrow k \longrightarrow 0.$$

Now suppose that $p > 2$. For our purposes it is not necessary to understand the Bockstein map. Rather we only want to know the extensions in $\mathrm{Ext}^2_{kG}(k,k) = H^2(G,k)$ that represents the Bockstein of an element of degree one. First notice that $H^1(G,k) \cong \mathrm{Hom}(G,k)$ by an argument that is almost precisely the same as that in the case that $p = 2$. In the same way as before, if $\zeta \in \mathrm{H}^1(G,k)$ then ζ is represented by a sequence

$$0 \longrightarrow k \longrightarrow k_H^{\uparrow G}/((\mathrm{Rad}\, kG)^2 \cdot k_H^{\uparrow G}) \longrightarrow k \longrightarrow 0,$$

for the corresponding maximal subgroup H. The Bockstein of γ is represented by the sequence

$$0 \longrightarrow k \longrightarrow k_H^{\uparrow G} \xrightarrow{(x-1)} k_H^{\uparrow G} \longrightarrow k \longrightarrow 0$$

Exercise 5.8. Suppose that $G = \langle x \mid x^9 = 1\rangle$ and $H = \langle x^3 \rangle$ is the unique maximal subgroup of G. Show that the sequence

$$0 \longrightarrow k \longrightarrow k_H^{\uparrow G} \xrightarrow{(x-1)} k_H^{\uparrow G} \longrightarrow k \longrightarrow 0$$

represents the zero element in $\mathrm{H}^2(G,k)$.

With the above analysis, we see that Serre's Theorem is equivalent to the following. The point is that the product of the cohomology elements is represented by the splice of the sequences.

Theorem 5.9. *Suppose that G is a p-group which is not elementary abelian. Then there is a sequence of maximal subgroups $H_1, \dots, H_n$ of G and an exact sequence*

$$\mathcal{E}: \quad 0 \longrightarrow k \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow k \longrightarrow 0$$

such that

(1) $C_i \cong k_{H_i}^{\uparrow G}$ *for $i = 1, \dots, n$, and*
(2) *the class of $\mathcal{E}$ in $\mathrm{Ext}^n_{kG}(k,k)$ is zero.*

Proof of the Main Theorem: Construction of the module $U \cong k \oplus V$. We do this part only for the special case that $G = D_8$, the dihedral group of order 8. The reader should be able to see how the general case would proceed. The notation is the same as in the last section. In particular, let $G = \langle x, y | x^2 = y^2 = (xy)^4 = 1\rangle$. Let $E = \langle x, (xy)^2\rangle$ and $F = \langle y, (xy)^2\rangle$ be the two elementary abelian subgroups of G. We have an exact sequence

$$\mathcal{E}: \qquad 0 \longrightarrow k \longrightarrow k_E^{\uparrow G} \longrightarrow k_F^{\uparrow G} \longrightarrow k \longrightarrow 0$$

which represents the element $\eta_1\eta_2 = 0$ in $\mathrm{H}^2(G,k) = \mathrm{Ext}^2_{kG}(k,k)$. This is the splice of the two sequence representing η_1 and η_2. Next we let $\mathcal{C}$ be the complex obtained by truncating the two ends (the copies of k) off of the ends of the sequence $\mathcal{E}$. Thus the complex $\mathcal{C}$ has the form

$$0 \longrightarrow \mathcal{C}_1 \longrightarrow \mathcal{C}_2 \longrightarrow 0$$

where $\mathcal{C}_1 \cong k_E^{\uparrow G}$ and $\mathcal{C}_2 \cong k_F^{\uparrow G}$. Then the homology of $\mathcal{C}$ is given by

$$\mathrm{H}_n\,\mathcal{C} \quad = \quad \begin{cases} k & \text{if} \quad n = 0,1, \\ 0 & \text{otherwise.} \end{cases}$$

Now let (P_*, ε) be a projective resolution of k:

$$(P_*, \varepsilon): \qquad \cdots \longrightarrow P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} k \longrightarrow 0.$$

We consider the complex $(P \otimes \mathcal{C})_*$

$$\cdots \longrightarrow \begin{matrix} P_2 \otimes \mathcal{C}_1 \\ \oplus \\ P_3 \otimes \mathcal{C}_0 \end{matrix} \longrightarrow \begin{matrix} P_1 \otimes \mathcal{C}_1 \\ \oplus \\ P_2 \otimes \mathcal{C}_0 \end{matrix} \longrightarrow \begin{matrix} P_0 \otimes \mathcal{C}_1 \\ \oplus \\ P_1 \otimes \mathcal{C}_0 \end{matrix} \longrightarrow P_0 \otimes \mathcal{C}_0 \longrightarrow 0$$

Note that the complex

$$\mathcal{C}_*^{(0)}: \qquad 0 \longrightarrow 0 \longrightarrow \mathcal{C}_0 \longrightarrow 0$$

is a subcomplex of $\mathcal{C}_*$. Also the complex

$$\mathcal{C}_*^{(1)}: \qquad 0 \longrightarrow \mathcal{C}_1 \longrightarrow 0 \longrightarrow 0$$

is a quotient complex of $\mathcal{C}_*$. That is, we have a exact sequences

$$0 \longrightarrow \mathcal{C}_*^{(0)} \longrightarrow \mathcal{C}_* \longrightarrow \mathcal{C}_*^{(1)} \longrightarrow 0$$

$$0 \longrightarrow (\mathcal{C}^{(0)} \otimes P)_* \longrightarrow (\mathcal{C} \otimes P)_* \longrightarrow (\mathcal{C}^{(1)} \otimes P)_* \longrightarrow 0$$

of complexes. Now we notice that

$$\mathrm{H}_*(\mathcal{C}^{(0)} \otimes P) \;=\; \begin{cases} \mathrm{H}_0(\mathcal{C}_*^{(0)}) \otimes \mathrm{H}_0(P_*) \cong k_E^{\uparrow G} & \text{if } n = 0 \\ \quad 0 & \text{otherwise} \end{cases}$$

and

$$\mathrm{H}_*(\mathcal{C}^{(1)} \otimes P) \;=\; \begin{cases} \mathrm{H}_1(\mathcal{C}_*^{(1)}) \otimes \mathrm{H}_0(P_*) \cong k_E^{\uparrow G} & \text{if } n = 1 \\ \quad 0 & \text{otherwise} \end{cases}$$

In particular, we see that $(\mathcal{C}^{(0)} \otimes P)_*$ is a projective resolution of $k_E^{\uparrow G}$, and $(\mathcal{C}^{(1)} \otimes P)_*$ is projective resolution of $k_F^{\uparrow G}$ except that it is shifted by one degree.

Now let $\Gamma_* = \Gamma_*(\mathcal{C} \otimes P)$ be the truncation of the complex $(\mathcal{C} \otimes P)_*$ at degree 2. That is, Γ_* has the form

$$\cdots \longrightarrow \begin{array}{c} P_3 \otimes \mathcal{C}_1 \\ \oplus \\ P_4 \otimes \mathcal{C}_0 \end{array} \longrightarrow \begin{array}{c} P_2 \otimes \mathcal{C}_1 \\ \oplus \\ P_3 \otimes \mathcal{C}_0 \end{array} \longrightarrow \begin{array}{c} P_1 \otimes \mathcal{C}_1 \\ \oplus \\ P_2 \otimes \mathcal{C}_0 \end{array} \longrightarrow 0.$$

Notice that for $n > 2$, $\mathrm{H}_n(\Gamma) = \mathrm{H}_n(\mathcal{C} \otimes P) = 0$. So Γ_* is a shift by two degrees of a projective resolution of $\mathrm{H}_2(\Gamma_*)$. Let $U = \Omega^{-2}(k) \otimes \mathrm{H}_2(\Gamma_*)$. It remains to prove that U has the properties that we want.

Proof of the Main Theorem: The filtration on U. A slight variation on our previous observation reveals that $\Gamma'_* = \Gamma_*(\mathcal{C}^{(0)} \otimes P)$ is a subcomplex of Γ_*, and moreover the quotient complex $(\Gamma/\Gamma')_*$ is isomorphic to $\Gamma''_* = \Gamma_*(\mathcal{C}^{(1)} \otimes P)$. So we have an exact sequence of complexes

$$0 \longrightarrow \Gamma'_* \longrightarrow \Gamma_* \longrightarrow \Gamma''_* \longrightarrow 0$$

and a corresponding long exact sequence on homology

$$0 \longrightarrow \mathrm{H}_2(\Gamma') \longrightarrow \mathrm{H}_2(\Gamma) \longrightarrow \mathrm{H}_2(\Gamma'') \longrightarrow 0$$

Next thing that we should observe is that $\mathrm{H}_2(\Gamma') \cong \Omega^2(k_F^{\uparrow G}) \oplus (\mathrm{proj})$ and that $\mathrm{H}_2(\Gamma'') \cong \Omega^1(k_E^{\uparrow G}) \oplus (\mathrm{proj})$. Hence we have an exact sequence

$$0 \longrightarrow \Omega^2(k_F^{\uparrow G}) \oplus (\mathrm{proj}) \longrightarrow \mathrm{H}_2(\Gamma) \longrightarrow \Omega^1(k_E^{\uparrow G}) \oplus (\mathrm{proj}) \longrightarrow 0$$

Then tensoring with $\Omega^{-2}(k)$ we get

$$0 \longrightarrow k_F^{\uparrow G} \oplus (\mathrm{proj}) \longrightarrow U \longrightarrow \Omega^{-1}(k_E^{\uparrow G}) \oplus (\mathrm{proj}) \longrightarrow 0$$

Finally, we observe that $\Omega^{-1}(k_E^{\uparrow G}) \cong (\Omega^{-1}(k_E))^{\uparrow G}$. Thus we have the filtration that we want.

Proof of the Main Theorem: Verifying that $U \cong k \oplus V$. The fact that $\eta_1\eta_2 = 0$ means that the sequence $\mathcal{E}$ is equivalent to the split sequence

$$0 \longrightarrow k \xrightarrow{\mathrm{Id}} k \xrightarrow{0} k \xrightarrow{\mathrm{Id}} k \longrightarrow 0$$

That in turn means that the complex is $\mathcal{C}_*$ is equivalent to a split complex $\mathcal{D}_*$ of the form

$$0 \longrightarrow k \xrightarrow{0} k \longrightarrow 0$$

by a sequence of chain maps that induce isomorphisms on homology. For the $\mathcal{D}_*$ we can prove the following.

Lemma 5.10. *Suppose that the complex $\Gamma_*(\mathcal{D} \otimes P)$ is as above. Then*

$$\mathrm{H}_2(\Gamma(\mathcal{D} \otimes P)) \cong \Omega^2(k) \oplus \Omega^1(k) \oplus (\mathrm{proj})$$

.

Proof. The point is that $(\mathcal{D}\otimes P)_*$ is the direct sum of $(\mathcal{D}^{(0)}\otimes P)_*\cong P_*$ and $(\mathcal{D}^{(1)}\otimes P)_*\cong P_*[1]$, where by $P_*[1]$ we mean the resolution P_* shifted one degree. Hence the homology, after applying the trunction, Γ is precisely as stated. □

It remains to prove that equivalences of complexes do not subtantially change the outcome of our process. To this end, suppose that we have a complex $\mathcal{U}_*$ and a chain map $\phi:\mathcal{U}_*\longrightarrow\mathcal{C}_*$ of the form

$$\begin{array}{ccccccccc}
\mathcal{U}_* & : & 0 & \longrightarrow & \mathcal{U}_1 & \longrightarrow & \mathcal{U}_0 & \longrightarrow & 0\\
\downarrow{\scriptstyle\phi} & & \downarrow & & \downarrow{\scriptstyle\phi_1} & & \downarrow{\scriptstyle\phi_0} & & \downarrow\\
\mathcal{C}_* & : & 0 & \longrightarrow & \mathcal{C}_1 & \longrightarrow & \mathcal{C}_0 & \longrightarrow & 0.
\end{array}$$

which induces an isomorphism on homology. Then we obtain a chain map

$$\phi\otimes 1:(\mathcal{U}\otimes P)_*\longrightarrow(\mathcal{C}\otimes P)_*$$

which also induces an isomorphism on homology. Unfortunately, it is not necessary that the induced chain map

$$\Gamma(\phi\otimes 1):\qquad \Gamma_*(\mathcal{U}\otimes P)\longrightarrow\Gamma_*(\mathcal{C}\otimes P)$$

induce an isomorphism on homology. However, it does induce an isomorphism in the stable category, and this is the way we should view these objects.

From a practical standpoint we need to alter to $(\mathcal{U}\otimes P)_*$ by adding a totally split complex of projective modules so that the resulting map $\phi:(\mathcal{U}\otimes P)_*\longrightarrow(\mathcal{C}\otimes P)_*$ is a surjection. This is accomplished as follows.

Suppose we have a chain map between two complexes $\theta_*:\mathcal{I}_*\longrightarrow\mathcal{J}_*$ such that in degree n, the module map $\theta_n:\mathcal{I}_n\longrightarrow\mathcal{J}_n$ fails to be surjective. Then we find a projective module Q with the property that there is a map $\alpha:Q\longrightarrow\mathcal{J}_n$ such that $(\theta_n,\alpha):\mathcal{I}_n\oplus Q\longrightarrow\mathcal{J}_n$ is surjective. It sufficient to choose Q to be a projective cover of the cokernel of θ_n. Now alter the complex $\mathcal{I}_*$ be taking direct sum with the exact complex $Q\longrightarrow Q$, and alter θ as in the diagram

$$\begin{array}{ccccccccc}
\cdots\longrightarrow\mathcal{I}_{n+1} & \longrightarrow & \mathcal{I}_n\oplus Q & \xrightarrow{(\partial,\mathrm{Id})} & \mathcal{I}_{n-1}\oplus Q & \longrightarrow & \mathcal{I}_{n-2}\longrightarrow\cdots\\
 & & \downarrow{\scriptstyle(\theta_n,\alpha)} & & \downarrow{\scriptstyle(\theta_{n-1},\partial'\alpha)} & & \\
\cdots & \longrightarrow & \mathcal{J}_n & \xrightarrow{\partial'} & \mathcal{J}_{n-1} & \longrightarrow & \cdots
\end{array}$$

Assuming that we have followed the above proceedure whenever necessary, we have now a surjective chain map

$$\phi':\mathcal{V}_*\longrightarrow(\mathcal{C}\oplus P)_*$$

where $\mathcal{V}_*\cong(\mathcal{U}\otimes P)_*\oplus\mathcal{Q}_*$ where $\mathcal{Q}_*$ is a totally split complex of projective modules. The important thing is that $\mathrm{H}_n(\Gamma(\mathcal{V}))\cong\mathrm{H}_n(\Gamma(\mathcal{U}\otimes P))\oplus\mathrm{H}_n(\Gamma(\mathcal{Q}))$, and the homology $\mathrm{H}_n(\Gamma(\mathcal{Q}))$ is zero except possibly in degree 2 where it is a projective module. In particular, we have that

$$\mathrm{H}_n(\Gamma(\mathcal{V})) \cong \mathrm{H}_n(\Gamma(\mathcal{U} \otimes P)) \oplus (\mathrm{proj})$$

In addition, the chain map $\phi' : \mathcal{V}_* \longrightarrow (\mathcal{C} \otimes P)_*$ induces an isomophism on homology. Let $\mathcal{K}_*$ denote the kernel of ϕ'. We see that the terms in $\mathcal{K}_*$ are projective modules. There is a long exact sequence on homology

$$\cdots \longrightarrow \mathrm{H}_2(\mathcal{K}) \longrightarrow \mathrm{H}_2(\mathcal{V}) \xrightarrow{\phi'_*} \mathrm{H}_2(\mathcal{C} \otimes P) \longrightarrow \mathrm{H}_1(\mathcal{K}) \longrightarrow \cdots$$

Because ϕ' induces an isomorphism on homology, we must have that $\mathrm{H}_n(\mathcal{K}) = \{0\}$ of all n. Thus, $\mathcal{K}_*$ is an exact complex of projective modules. It follows that $\mathrm{H}_n(\Gamma(\mathcal{K}))$ is zero except possibly in degree 2 where it might be a projective module. Hence by the same arguments as before, we have that

$$\mathrm{H}_2(\mathcal{V}) \cong \mathrm{H}_2(\Gamma(\mathcal{C} \otimes P)) \oplus (\mathrm{proj}) \cong \mathrm{H}_2(\Gamma(\mathcal{U} \otimes P)) \cong \Omega^2(k) \oplus \Omega(k) \oplus (\mathrm{proj})\,.$$

So finally, we can see that

$$U = \Omega^{-2}(\mathrm{H}_2(\Gamma(\mathcal{U} \otimes P))) \cong k \oplus \Omega^{-1}(k) \oplus (\mathrm{proj}),$$

as was to be proved.

Remarks on the general proof. In a full proof of the theorem we would follow the lines of what we have done for the example of the dihedral group. Usually, the filtration would be more complicated because the complex $\mathcal{C}_*$ would have more than two terms. So the filtration would have to be demonstrated by a sequence of exact sequences of complexes.

In addition, the argument of the example would be only the induction step in the argument of a general proof. The point is that the arugment only shows that the trivial module k is a direct summand of a module which can be filtered by modules induced from maximal subgroups of G. In the case that $G \cong D_8$, the maximal subgroups are elementary abelian and the proof is complete. In a more general situation, the maximal subgroup H might not be elementary abelian. Then we must assume by induction that the theorem is true for the maximal subgroup and replace that section of the complex by a module which is filtered by modules induced from the elementary abelian subgroups of H. Each time this is done the module V in the theorem, aquires new direct summands. For a complete version of the proof see the original paper [C2] or the book [10].

6. Quillen's Dimension Theorem

The object of this section is to prove a form of Quillen's Dimension Theorem and see some of its consequences. The statement of the theorem concerns varieties or maximal ideal spectra of rings. But the proof is really about the structure of the ring. Indeed, in the proof we see something of the connection between the ring structure and the structure of its variety.

To state and prove the main theorem of this section, we adopt the notation of the last section. From the statement of Theorem 5.1 we know that there is a natural number τ, a finite dimensional kG-module U and a sequence of elementary abelian subgroups $E_1, \dots, E_\tau$ such that there is a filtration

$$0 = L_0 \subseteq L_1 \subseteq \dots L_\tau = k \oplus V$$

where for each i, $L_i/L_{i-1} \cong W_i^{\uparrow G}$ for some kE_i-module W_i. The proof of some of the corollaries involved tensoring everything in this system with another module M and deducing that M is a direct summand of a module that is filtered by modules that are the inductions of restrictions of M to elementary abelian subgroups. It is noteworthy that the sequence of subgroups does not change from the sequence $E_1, \dots, E_n$ of elementary abelian p-subgroups in the theorem. Hence, we can consider these items as invariants of the group G, although their selection is certainly not unique. What follows is the theorem that we need.

Theorem 6.1. *Suppose that we have homogeneous elements $\zeta_1, \dots, \zeta_\tau \in \mathrm{H}^*(G,k)$ with the property that $\mathrm{res}_{G,E_i}(\zeta_i) = 0$. Then $\zeta_\tau \cdots \zeta_2\zeta_1 = 0$.*

Proof. We know that there exists a kG-module V such that $U = k \oplus V$ has a filtration

$$0 \subseteq L_0 \subseteq L_1 \subseteq \dots \subseteq L_\tau = U$$

where $L_i/L_{i-1} \cong W_i^{\uparrow G}$ for some kE_i-module W_i. Now for any m, $\mathrm{H}^m(G,k) = \mathrm{Ext}^m_{kG}(k,k)$ is embedded in $\mathrm{Ext}^m_{kG}(U,U)$ as a direct summand. So we may identify ζ_i with its inclusion into $\mathrm{Ext}^{m_i}_{kG}(U,U)$, where m_i is the degree of ζ_i. Moreover, because the sums are direct, if we show that the product $\zeta_n \cdots \zeta_1$ is zero as an element of $\mathrm{Ext}^*_{kG}(U,U)$, then we are done.

For each i we have an exact sequence

$$0 \longrightarrow L_{i-1} \xrightarrow{j_i} L_i \xrightarrow{q_i} W_i^{\uparrow G} \longrightarrow 0$$

The long exact sequence in cohomology, is natural with respect to multiplication by cohomology elements. So we have a diagram

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \mathrm{Ext}^r_{kG}(U, L_{i-1}) & \xrightarrow{(j_i)_*} & \mathrm{Ext}^r_{kG}(U, L_i) & \xrightarrow{(q_i)_*} & \mathrm{Ext}^r_{kG}(U, W_i^{\uparrow G}) & \longrightarrow \cdots \\
& \downarrow \zeta_i & & \downarrow \zeta_i & & \downarrow \zeta_i & \\
\cdots \longrightarrow & \mathrm{Ext}^{r+m_i}_{kG}(U, L_{i-1}) & \xrightarrow{(j_i)_*} & \mathrm{Ext}^{r+m_i}_{kG}(U, L_i) & \xrightarrow{(q_i)_*} & \mathrm{Ext}^{r+m_i}_{kG}(U, W_i^{\uparrow G}) & \longrightarrow \cdots
\end{array}$$

where the vertical maps are right multiplication by ζ_i. In addition, there is a commutative diagram

$$\begin{array}{ccc}
\mathrm{Ext}^r_{kG}(U, W_i^{\uparrow G}) & \xrightarrow{\cong} & \mathrm{Ext}^r_{kE_i}(U, W_i) \\
\downarrow \zeta_i & & \downarrow \mathrm{res}_{G,E_i}(\zeta_i) \\
\mathrm{Ext}^{r+m_i}_{kG}(U, W_i^{\uparrow G}) & \xrightarrow{\cong} & \mathrm{Ext}^{r+m_i}_{kE_i}(U, W_i)
\end{array}$$

where the horizontal maps are isomorphisms. The horizontal isomorphisms are certified by the Eckmann-Shapiro Lemma (see below). By the hypothesis of the theorem, we know that $\mathrm{res}_{G,E_i}(\zeta_i) = 0$. Therefore we must have that multiplication by ζ_i is zero on $\mathrm{Ext}^r_{kG}(U, W_i^{\uparrow G})$.

Now we can complete the proof of the theorem. Consider first the case $i = \tau$ (so $L_i = U$) and $r = 0$. From the two diagrams above, we conclude that $(q_\tau)_*(\mathrm{Id}_U \cdot \zeta_\tau) = 0$ and hence $\mathrm{Id}_U \cdot \zeta_\tau = \zeta_\tau = (j_\tau)_*(\phi_\tau)$ for some $\phi_\tau \in \mathrm{Ext}^{m_\tau}_{kG}(U, L_{\tau-1})$. Now assume by induction that $\zeta_\tau \cdots \zeta_{i+1} = (j_\tau)_* \cdots (j_{i+1})_*(\phi_{i+1})$ for some ϕ_{i+1} in $\mathrm{Ext}^r_{kG}(U, L_i)$, where $r = m_{i+1} + \ldots + m_\tau$. Then we have that

$$\zeta_\tau \cdots \zeta_i = ((j_\tau)_* \cdots (j_{i+1})_* \phi_{i+1})\zeta_i = (j_\tau)_* \cdots (j_{i+1})_*(\phi_{i+1}\zeta_i).$$

It is also true that $\phi_{i+1}\zeta_i = (j_i)_*\phi_i$ for some element $\phi_i \in \mathrm{Ext}^{r+m_i}_{kG}(U, L_{i-1})$. If we persist along this path, we eventually prove that

$$\zeta_\tau \cdots \zeta_1 = (j_\tau)_* \cdots (j_1)_*(\phi_1)$$

for some $\phi_1 \in \mathrm{Ext}^s_{kG}(U, L_0)$. As $L_0 = \{0\}$, the product is 0. This finishes the proof. □

Exercise 6.2. Suppose that M is a kG-module and N is a kH-module where H is a subgroup of G. Prove that

$$\mathrm{Hom}_{kG}(M, N^{\uparrow G}) \;\cong\; \mathrm{Hom}_{kH}(M_H, N)$$

and also that

$$\mathrm{Ext}^n_{kG}(M, N^{\uparrow G}) \;\cong\; \mathrm{Ext}^n_{kH}(M_H, N)$$

for any n. Hint: For $f : M_H \longrightarrow N$ a kH-homomorphism, let $\psi(f) : M \longrightarrow N^{\uparrow G}$ be defined by $\psi(f)(m) = \sum_{gH} g \otimes f(g^{-1}m)$. Likewise, for $f : M \longrightarrow N^{\uparrow G}$ define $\theta(f) : M_H \longrightarrow N$ by $\theta(f)(\sum_{gH} g \otimes m_g) = f(m_1)$. The sums should be over a complete set of representative of the left cosets of H in G. And we always assume that the representative of the identity coset H is the element $1 \in G$.

Corollary 6.3. *Suppose that I is the ideal in $\mathrm{H}^*(G, k)$ consisting of all elements ζ having the property that the restriction $\mathrm{res}_{G,E}(\zeta) = 0$ for all elementary abelian p-subgroups E of G. Then $I^\tau = \{0\}$.*

That is the set of all elements in $\mathrm{H}^*(G, k)$ which vanish on restriction to every elementary abelian subgroup of G is a nilpotent ideal. Because, $\mathrm{H}^*(G, k)$ is finitely generated as a k-algebra, it is noetherian as a ring and its Jacobson radical is nilpotent. The ideal I in the corollary is clearly in the Jacobson radical. If E is an elementary abelian p-subgroup of G, then $\mathrm{H}^*(E, k)/\operatorname{Rad}(\mathrm{H}^*(E, k))$ is a polynomial ring. Consequently we have the following.

Corollary 6.4. *The Jacobson radical of $\mathrm{H}^*(G, k)$ is the ideal consisting of all ζ such that $\mathrm{res}_{G,E}(\zeta) \in \operatorname{Rad}(\mathrm{H}^*(E, k))$ for all elementary abelian p-subgroups E of G.*

Actually we can say more. Let $E_1, \ldots, E_r$ be a complete set of representatives of the conjugacy classes of maximal elementary abelian p-subgroups of G. Now if E is any elementary abelian p-subgroup then E is conjugate to a subgroup of some E_i and the restriction from G to E factors through the restriction map from G to E_i. Consequently, the Jacobson radical can be characterized as the intersection of the kernels of the composition of the restriction map to E_i with the quotient map onto $\mathrm{H}^*(E_i,k)/\operatorname{Rad}(\mathrm{H}^*(E_i,k))$. So let $\mathfrak{p}_i$ be the kernel of the composition

$$\mathrm{H}^*(G,k) \longrightarrow \mathrm{H}^*(E_i,k) \longrightarrow \mathrm{H}^*(E_i,k)/\operatorname{Rad}(\mathrm{H}^*(E_i,k)).$$

Then we can prove the following.

Corollary 6.5. *The ideals $\mathfrak{p}_i$ are the minimal primes in $\mathrm{H}^*(G,k)$. Moreover*

$$\operatorname{Rad}(\mathrm{H}^*(G,k)) \;=\; \cup_{i=1}^{r}\mathfrak{p}_i.$$

The ideals $\mathfrak{p}_i$ are prime because the image of $\mathrm{H}^*(G,k)$ in $\mathrm{H}^*(E_i,k)/\operatorname{Rad}(\mathrm{H}^*(E_i,k))$ is an integral domain. It takes a bit more work to show that these are the minimal primes. We leave this to the reader to ponder.

Next we want to restate all of this in terms of the geometry. For this we need some commutative algebra. In particular, the following is essentially Theorem 9.3 and Exercise 9.3 of Matsumora's book [Mats].

Theorem 6.6. *Let A and B be finitely generated graded commutative algebras over a field k, with $A \subseteq B$. We assume that B is finitely generated as a module over A. Then for any prime ideal $\mathfrak{p}$ in A we have that*

(1) *There exists a prime ideal of B lying over $\mathfrak{p}$ and*
(2) *B has only a finite number of prime ideals that lie over $\mathfrak{p}$.*

Quillen's Theorem is the following.

Theorem 6.7. *Suppose that $E_1,\ldots,E_r$ are a complete set of representatives of the conjugacy classes of maximal elementary abelian p-subgroups of the finite group G. Then*

$$V_G(k) = \bigcup_{i=1}^{t} \operatorname{res}^*_{G,E_i}(V_{E_i}(k))$$

In particular, each of the maps $\operatorname{res}^*_{G,E_i}$ is finite-to-one on varieties and the images $\operatorname{res}^*_{G,E_i}(V_{E_i}(k))$ are the irreducible components of $V_G(k)$.

Proof. If $\alpha \in V_G(k)$ is a maximal ideal of the ring $\mathrm{H}^*(G,k)$, then one of the minimal primes $\mathfrak{p}_i$ must be contained in α for some i. Therefore, $\operatorname{res}^*_{G,E_i}(\alpha)$ is a maximal ideal within the image $\operatorname{res}_{G,E_i}(\mathrm{H}^*(G,k))$ in $\mathrm{H}^*(E_i,k)$. Because $\mathrm{H}^*(E_i,k)$ is a finitely generated module over $\operatorname{res}_{G,E_i}(\mathrm{H}^*(G,k))$, the previous theorem applies. Hence, there are finitely many prime ideals $\gamma_1,\ldots,\gamma_r$ in $V_{E_i}(k)$ that lie over α. Finally, since $\mathrm{H}^*(E_i,k)/\gamma_j$ is finitely generated as a module over the field $\mathrm{H}^*(G,k)/\alpha$, then each γ_j must be a maximal ideal. □

7. Properties of support varieties

In this section, we explore and justify a couple of the properties of support varieties that we want for the next section. In view of the fact that we have omitted the proof of so many other things, the proofs of these properties can be considered to be complete. Our aim is not a complete verification of the results, but rather we want to give some flavor of the type of proofs that are possible in the area.

Recall that the variety of kG-module M is the variety $V_G(M) = V_G(J(M)) \subseteq V_G(k)$ of the ideal $J(M)$ which is the annihilator in $\mathrm{H}^*(G,k)$ of the cohomology ring $\mathrm{Ext}^*_{kG}(M,M)$ of M. That is, it is the closed set of all maximal ideals in $\mathrm{H}^*(G,k)$ that contain $J(M)$. Using methods that are very similar to those of the last section, we can prove the follow theorem. This theorem was first proved by Alperin and Evens [AE2] and independently by Avrunin [Av].

Theorem 7.1. *If M is a finitely generated kG-module, then*

$$V_G(M) = \bigcup_{E \in \mathcal{A}} \mathrm{res}^*_{G,E}(V_E(M))$$

where $\mathcal{A}$ is the set of elementary abelian p-subgroups of G.

The proof follows the same ideas as that of Quillen's Theorem 6.7. We need to show that a homogeneous element ζ in $\mathrm{H}^*(G,k)$ annihilates $\mathrm{Ext}^*_{kG}(M,M)$ if and only if its restriction every elementary abelian subgroup E of G annihilates the cohomology of M_E.

The theorem shows that we can measure the variety of a module M by looking at its restrictions. An example of a place where we might want to do this is in the computation of the varieties of the modules L_ζ. The modules L_ζ are defined as follows. Let ζ be a nonzero element of $\mathrm{H}^m(G,K)$ and suppose that $\hat{\zeta} : \Omega^m(k) \longrightarrow k$ is a cocycle representing ζ. The module L_ζ is defined to be the kernel of $\hat{\zeta}$. To be consistent, we let $L_\zeta = \Omega^m(k) \oplus \Omega(k)$ in the case that ζ is the zero element of $\mathrm{H}^m(G,k)$. In any event, we have an exact sequence

$$E_\zeta : \qquad 0 \longrightarrow L_\zeta \longrightarrow \Omega^m(k) \oplus (\mathrm{proj}) \xrightarrow{\hat{\zeta}} k \longrightarrow 0.$$

If $\zeta = 0$, then the projective direct summand (proj) can be assumed to be the projective cover of k. Otherwise, (proj) is the zero module. The support variety of L_ζ is given in the following.

Proposition 7.2. *Suppose that $\zeta \in \mathrm{H}^m(G,k)$. Assume that m is even if $p > 2$. Then $V_G(L_\zeta) = V_G(\zeta)$ is the set of all maximal ideals that contain ζ.*

Proof. Let α be any element of $V_G(k)$. Then there exists an elementary abelian p-subgroup $E = \langle x_1, \dots, x_r \rangle$, such that α contains the kernel of the restriction to kE. That is, the homomorphism $\overline{\alpha} : \mathrm{H}^*(G,k) \longrightarrow k$ factors through the restriction to $\mathrm{H}^*(E,k)$. Now we use the rank variety of the module (see 3.5). We know that there is a cyclic shifted subgroup $U = \langle u_\alpha \rangle$ where

$$u_\alpha \;=\; \sum_{i=1}^{r} \alpha_i (x_i - 1)$$

such that $\overline{\alpha}$ factors through restriction to U. If $\alpha \notin V_G(\zeta)$, then $\mathrm{res}_{G,U}(\zeta)$ is not nilpotent, and hence it is not zero. By assumption, if $p > 2$, then m must be even. Therefore, whether p is even or odd, $\Omega^m(k_U) \cong k$, and $\Omega^m(k)_{\downarrow U} \cong k_U \oplus (\mathrm{proj})$. Thus, the sequence E_ζ must split on restriction to U because $\mathrm{res}_{G,U}(\zeta) \neq 0$. Therefore L_ζ is free as a kU-module and $\alpha \notin V_G(L_\zeta)$. If, on the other hand, $\alpha \notin V_G(L_\zeta)$, then $(L_\zeta)_{\downarrow U}$ is free, $\mathrm{res}_{G,U}(\zeta) \neq 0$ and $\alpha \notin V_G(\zeta)$. $\square$

Next we turn our attention to the connectedness theorem. It says that we can under some circumstances tell if a module is indecomposable by knowing its variety. For the proof of this theorem, we need the Tensor Product Theorem for support varieties. We assume that it is true without proof.

Theorem 7.3. *Suppose that M and N are kG-modules. Then*

$$V_G(M \otimes N) \quad = \quad V_G(M) \cap V_G(N).$$

We also need a theorem on the annihilation of cohomology.

Proposition 7.4. *Let $\zeta \in \mathrm{H}^n(G,k)$ and let M be a finitely generated kG-module. Then ζ annihilates $\mathrm{Ext}^*_{kG}(M,M)$ if and only if*

$$M \otimes L_\zeta \cong \Omega(M) \oplus \Omega^n(M) \oplus (\mathrm{proj})\,.$$

Proof. The element ζ as an element of $\mathrm{Ext}^n_{kG}(k,k) \cong \mathrm{Ext}^1_{kG}(\Omega^{n-1}(k),k)$ is represented by the sequence

$$\mathcal{E}_\zeta : \quad 0 \longrightarrow k \longrightarrow \Omega^{-1}(L_\zeta) \longrightarrow \Omega^{n-1}(k) \longrightarrow 0.$$

This can be seen from the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & L_\zeta & = & L_\zeta & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Omega^n(k) & \longrightarrow & P_{n-1} & \longrightarrow & \Omega^{n-1}(k) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \zeta'} & & \downarrow & & \| & & \\
0 & \longrightarrow & k & \longrightarrow & \Omega^{-1}(L_\zeta) & \longrightarrow & \Omega^{n-1}(k) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where P_{n-1} is the $n-1$ term in the projective resolution of k and ζ' is a cocycle representating ζ. Then we have that $\zeta \cdot \mathrm{Id}_M \in \mathrm{Ext}^n_{kG}(M,M) \cong \mathrm{Ext}^1_{kG}(\Omega^{n-1}(M),M)$ is represented by the sequence

$$\mathcal{E}_\zeta \otimes M : \quad 0 \longrightarrow M \longrightarrow \Omega^{-1}(L_\zeta) \otimes M \longrightarrow \Omega^{n-1}(k) \otimes M \longrightarrow 0.$$

If ζ annihilates the cohomology of M, then $\zeta \cdot \mathrm{Id}\, M = 0$ and the sequence $\mathcal{E}_\zeta \otimes M$ splits. Hence the middle term

$$\Omega^{-1}(L_\zeta) \otimes M \cong \Omega^{-1}(L_\zeta \otimes M) \oplus (\mathrm{proj})$$

is the direct sum of the two end terms. Now we need only translate everything by Ω to complete the necessary condition of the proposition. Conversely if $M \otimes L_\zeta \cong \Omega(M) \oplus \Omega^n(M) \oplus (\mathrm{proj})$, then the sequence splits and $\zeta \cdot \mathrm{Id}\, M = 0$. □

Actually in the above analysis we also need to know if the middle term of an exact sequence is the direct sum of the two end terms, then the sequence splits. That is, we need the following.

Exercise 7.5. Suppose that R is a finite dimensional algebra over a field k and that

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is an exact sequence of finitely generated R-modules such that $B \cong A \oplus C$. Prove that the sequence splits. **Hint:** Show by a dimension argument that the connecting homomorphism δ in the long exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(C,A) \longrightarrow \mathrm{Hom}_R(C,B) \longrightarrow \mathrm{Hom}_R(C,C) \xrightarrow{\delta} \mathrm{Ext}^1_R(C,A) \longrightarrow \cdots$$

is the zero map. Use this fact to get a splitting of the sequence.

Now we turn to the connectedness theorem. One implication of the theorem is that the variety of an indecomplable module is connected as a projective variety. That is, if we have two subvarieties V_1 and V_2 in $V_G(k)$, the statement that $V_1 \cap V_2 = \{0\}$ is the same as saying that the projective varieties $\overline{V}_1$ and $\overline{V}_2$ intersect in the empty set.

Theorem 7.6. *Suppose that M is a finitely generated kG-module with the property that $V_G(M) = V_1 \cup V_2$ where V_1 and V_2 are nonzero closed subvarieties such that $V_1 \cap V_2 = \{0\}$. Then $M \cong M_1 \oplus M_2$ where $V_G(M_1) = V_1$ and $V_G(M_2) = V_2$.*

Proof. First, notice that we can assume that neither V_1 nor V_2 is zero, and proceed by induction on the sum of the dimensions of V_1 and V_2. We should observe in the course of the proof that the minimal case in which both V_1 and V_2 have dimension 1 is covered in the argument that follows.

For some n there exists an element $\zeta \in \mathrm{H}^n(G,k)$ with the properties that $V_1 \subseteq V_G(\zeta)$ and $\dim(V_2 \cap V_G(\zeta)) \leq \dim V_2$. That is, ζ can be chosen to be in the ideal that defines V_1 but not in the ideal of any component of V_2. Likewise choose $\gamma \in \mathrm{H}^m(G,k)$ such that $V_2 \subseteq V_G(\gamma)$ and $\dim(V_1 \cap V_G(\gamma)) \leq \dim V_1$. We use the symbol ζ to denote a cocycle $\zeta : \Omega^n(k) \longrightarrow k$ that represents the cohomology class ζ. Let $\Omega^n(\gamma) : \Omega^{m+n}(k) \longrightarrow \Omega^n(k)$ be a representative of the class of γ in $\underline{\mathrm{Hom}}_{kG}(\Omega^{m+n}(k), \Omega^n(k)) \cong \mathrm{H}^m(G,k)$. Then the composition $\zeta \circ \Omega^n(\gamma)$ represents the cup product $\zeta\gamma$.

Hence we have the diagram

$$
\begin{array}{cccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
\mathcal{E}: & 0 & \longrightarrow & \Omega^n(L_\gamma) & \longrightarrow L_{\zeta\gamma} \oplus Q & \longrightarrow & L_\zeta & \longrightarrow & 0 \\
 & & & \| & \downarrow & & \downarrow & & \\
 & 0 & \longrightarrow & \Omega^n(L_\gamma) & \longrightarrow \Omega^{m+n}(k) \oplus Q & \xrightarrow{\Omega^n(\gamma)} & \Omega^n(k) & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \zeta\gamma} & & \downarrow{\scriptstyle \zeta} & & \\
 & & & & k & = & k & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

with exact rows and columns. Here Q is a projective module which is added to $\Omega^{m+n}(k)$ to be certain that the map $\Omega^n(\gamma)$ is surjective. Often it will be the zero module.

Now notice that $V_G(M) \subseteq V_G(L_{\zeta\gamma}) = V_G(\zeta\gamma) = V_G(\zeta) \cup V_G(\gamma)$. Consequently, some power $\zeta^t\gamma^t$ of $\zeta\gamma$ annihilates the cohomology of M. Without loss of generality, we replace ζ by ζ^t and γ by γ^t so that $\zeta\gamma$ annihilates $\mathrm{Ext}^*_{kG}(M, M)$.

Now consider the tensor product sequence $\mathcal{E} \otimes M$. We should observe that $V_G(L_\zeta \otimes M) = V_1 \cup (V_2 \cap V_G(\zeta))$. Then by induction, $L_\zeta \otimes M \cong X_1 \oplus X_2$, where $V_G(X_1) = V_1$ and $V_G(X_2) = V_2 \cap V_G(\zeta)$. Likewise, $\Omega^n(L_\gamma) \otimes M \cong Y_1 \oplus Y_2$ where $V_G(Y_1) = V_1 \cap V_G(\gamma)$ and $V_G(Y_2) = V_2$. Hence the sequence $\mathcal{E} \otimes M$ has the form

$$0 \to Y_1 \oplus Y_2 \to \Omega^{m+n}(M) \oplus \Omega(M) \oplus (\mathrm{proj}) \to X_1 \oplus X_2 \to 0.$$

To complete the proof, notice that $\mathrm{Ext}^1_{kG}(X_1, Y_2) = 0$ and also that $\mathrm{Ext}^1_{kG}(X_2, Y_1) = 0$. That is, Y_2 can not extend X_1 and Y_1 can not extend X_2. The only conclusion possible is that $\mathcal{E} \otimes M$ is the direct sum of two exact sequences, an extension N_1 of X_1 by Y_1 and another extension N_2 of X_2 by Y_2. So in the middle term we have

$$\Omega^{m+n}(M) \oplus \Omega(M) \oplus (\mathrm{proj}) \cong N_1 \oplus N_2.$$

where $V_G(N_1) \subseteq W_1$ and $V_G(N_2) \subseteq W_2$. The Krull-Schmidt Theorem guarantee that $\Omega(M)$ and $\Omega^{m+n}(M)$ decompose as desired, and so does M. $\square$

8. The rank of the group of endotrivial modules

Recall that a kG-module is an endotrivial module if $M^* \otimes M \cong \mathrm{Hom}_k(M, M) \cong k \oplus (\mathrm{proj})$. It is not difficult to see that if M is an endotrivial module, then $M \cong M_0 \oplus (\mathrm{proj})$ where M_0 is an indecomposable endotrivial module. Hence we can put an equivalence relation on the set of endotrivial modules, by saying that M is equivalent to N if and only if there exist projective modules P and Q such that $M \oplus P \cong N \oplus Q$. Let $[M]$ denote the equivalence class of an endotrivial module M. The group of endotrivial modules consists of the set $T(G)$ of equivalence classes of endotrivial modules, with the operation

$$[M] + [N] = [M \otimes N].$$

This is an abelian group and the inverse of a class $[M]$ is the class of the dual $[M^*]$.

There are two important theorems which we need to assume here. The first result which got the subject was proved by Dade [13] more than 25 years ago.

Theorem 8.1. *Suppose that G is an abelian p-group and that M is an endotrivial kG-module. Then $M \cong \Omega^n(k) \oplus \text{(proj)}$ for some n.*

We know that if G is a cyclic p-group then $\Omega^2(k) \cong k$. Also, if $|G| = 2$ then $\Omega(k) \cong k$. So we get immediately that

Corollary 8.2. *Let G be an abelian p-group then*

$$T(G) = \begin{cases} \{0\} & \textit{if } |G| = 2, \\ \mathbb{Z}/2\mathbb{Z} & \textit{if } G \textit{ is cyclic and } |G| > 2 \\ \mathbb{Z} & \textit{otherwise.} \end{cases}$$

The other theorem that we want was originally due to Puig [20], but it also follows from the complete classification [9] of endotrivial modules.

Theorem 8.3. *Let G be any group. Let $\mathcal{A}$ be a complete set of representatives of the conjugacy classes of maximal elementary abelian p-subgroups of G. Then the kernel of the product of the restriction maps*

$$\prod_{E \in \mathcal{A}} \text{res}^*_{G,E} : \qquad T(G) \longrightarrow \prod_{E \in \mathcal{A}} T(E)$$

is a finite group.

For the proof of our theorem, we also want Alperin's result [1] on the fusion of elementary abelian subgroups.

Theorem 8.4. *Suppose that E and F are two elementary abelian subgroups of G both having rank at least 3. Let M be an endotrivial kG-module and assume that $M_E \cong \Omega^a(k)$ while $M_F \cong \Omega^b(k)$. Then $a = b$.*

From now on we assume that the rank of G is at least 2. Let $E_1, \dots, E_n$ be maximal elementary abelians subgroups of G with the following properties.

- If the rank of G is 2 then $E_1, \dots, E_n$ is a complete set of representatives of the conjugacy classes of maximal elementary abelian p-subgroups of G.
- If the rank of G is greater than 2, then $E_1, \dots, E_{n-1}$ is a complete set of representatives of the conjugacy classes of maximal elementary abelian p-subgroups rank 2 of G and E_n has rank larger than 2.

Define the type of an endotrivial module M to be the n-tuple $a_1, \dots, a_n$ where $M_{E_i} \cong \Omega^{a_i}(k) \oplus \text{(proj)}$. From the theroems of Alperin and Puig we get the following.

Proposition 8.5. *The map $T(G) \longrightarrow \mathbb{Z}^n$ which takes $[M]$ to Type(M) has finite kernel.*

Consequently, the image of the Type map is a torsion free subgroup. of $\prod T(E_i)$. Our objective is to prove that the cokernel of the Type map is finite. In other words we want the following. This theorem was first proved for p-groups by Alperin [1], where he showed that we could use relative syzygies to construct endotrivial modules having sufficiently diverse types. The proof we give here is very different. For one thing, we do not have to assume that G is a p-group. It shows that set of generators for a torsion free subgroup of $T(G)$ having the right rank can be constructed by carving up the modules $\Omega^n(k)$ for certain n.

Theorem 8.6. *Assume that the p-rank of G is at least 2. Then the torsion-free rank of the group $T(G)$ is the number n.*

Proof. Notice that if $n = 1$, then there is nothing more to prove. So for the remainder of the proof assume that $n > 1$.

By Quillen's Dimension Theorem, we know that

$$V_G(k) = \bigcup_{E \in \mathcal{C}} \text{res}^*_{G,E}(V_E(k)).$$

If E has p-rank two, then $V_E(k)$ is a plane, and the intersection of $\text{res}^*_{G,E}(V_E(k))$ with $\text{res}^*_{G,F}(V_F(k))$, for F not conjugate to E, is a subvariety W of dimension one. In fact, the intersection W is precisely the subvariety $\text{res}^*_{G,Z}(V_Z(k))$ where Z is a subgroup of order p in the center of some Sylow p-subgroup of G. Thus we can get that for some $m > 0$, there exists an element $\zeta \in \text{H}^m(G, k)$ with the property that $V_G(\zeta)$, the set of all maximal ideals containing ζ, intersects W transversely. It is sufficient here to choose the element ζ so that $\text{res}_{G,Z}(\zeta)$ is not nilpotent, or equivalently, so that $\text{res}_{G,Z}(\zeta) \neq 0$ and that m is even if $p > 2$. Assume that such an element has been chosen.

Let $\zeta' : \Omega^m(k) \to k$ be a cocycle that represents ζ. The kernel of ζ', L_ζ, is a module having support variety $V_G(L_\zeta) = V_G(\zeta)$. Then, the support variety is disconnected. That is, we have that

$$V_G(L_\zeta) = V_1 \cup V_2 \cup \cdots \cup V_n$$

where

$$V_i = \bigcup_{E \in \mathcal{C}_i} \text{res}^*_{G,E}(V_E(L_\zeta)).$$

Notice here that when $i < n$ and in any case that E_i is a maximal elementary abelian group of rank 2, we have that $V_i = \text{res}^*_{G,E_i}(V_{E_i}(L_\zeta)) = \text{res}^*_{G,E_i}(V_{E_i}(\text{res}_{G,E_i}(\zeta)))$. Moreover, because $V_G(L_\zeta)$ is transverse to W, it is necessary that $V_i \cap V_j = \{0\}$. Consequently, L_ζ decomposes as

$$L_\zeta = L_1 \oplus L_2 \oplus \cdots \oplus L_n$$

where $V_G(L_i) = V_i$.

For any $i < n$, let $V_i' = \cup_{j \neq i} V_j$ and let $L_i' = \oplus_{j \neq i} L_j$. Then we have a diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & L_i' & = & L_i' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & L_\zeta & \longrightarrow & \Omega^m(k) & \xrightarrow{\zeta'} & k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & L_i & \longrightarrow & N_i & \longrightarrow & k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

where N_i is the pushout.

We claim that N_i is an endotrivial module. For the proof it is only necessary to show that the restriction of N_i to any maximal elementary abelian p-subgroup E of G is an endotrivial module. Consider first the restriction to E_i. Because V_i', which is the support variety of L_i', intersects $\operatorname{res}^*_{G,E_i}(V_E(k))$ in the set $\{0\}$ we conclude that $(L_i')_{E_i}$ is a projective kE-module. Hence on restriction to the subgroup E_i, the middle column splits, and $(N_i)_{E_i} \cong \Omega^m(k) \oplus (\text{proj})$. Thus, $(N_i)_{E_i}$ is an endotrivial module. On the other hand, suppose we consider E_j for $j \neq i$. Then by a similar argument, L_i is projective on restriction to E_j. So this time the bottom row in the diagram splits on restriction to E_j, and we have that $(N_i)_{E_j} \cong k \oplus (\text{proj})$.

Hence, we have constructed a collection $N_1, \ldots, N_{n-1}$ of endotrivial modules. Moreover, we know the restriction of any N_i to any maximal elementary abelian p-subgroup of G. In particular, we know that the Type of N_i is $(0, \ldots, 0, m, 0, \ldots, 0)$. Therefore, we have that the classes of the modules $\Omega(k), N_1, \ldots, N_{n-1}$ generate a torsion-free subgroup of $T(G)$ that has rank n. This proves the theorem. $\square$

References

[1] J. L. Alperin, *A construction of endo-permutation modules*, J. Group Theory **4** (2001), 3–10.

[AE1] J. L. Alperin and L. Evens, *Representations, resolutions and Quillen's dimension theorem*, J. Pure Appl. Algebra **22**(1981), 1-9.

[AE2] J. L. Alperin and L. Evens, *Varieties and elementary abelian subgroups*, J. Pure Appl. Algebra **26**(1982), 221-227.

[Av] G. S. Avrunin, *Annihilators of cohomology modules*, J. Algebra **69**(1981), 150-154.

[AS] G. S. Avrunin and L. Scott, *Quillen stratification for modules*, Invent. Math. **66**(1982), 277-286.

[2] D. J. Benson, "Representations and Cohomology I, II", Cambridge Univ. Press, 1991.

[3] D. J. Benson and J. F. Carlson, *Periodic modules with large period*, Quart. J. Math., **43**(1992), 283-296.

[4] D. J. Benson and J. F. Carlson, *The cohomology of extraspecial groups*, Bull. London Math. Soc. **24** (1992), 209–235.

[5] S. Bouc and J. Thévenaz, *The group of endo-permutation modules*, Invent. Math. **139** (2000), 275–349.

[C1] J. F. Carlson, *The varieties and the cohomology ring of a module* J. Algebra **85**(1983), 104-143

[C2] J. F. Carlson, *cohomology and Induction from elementary abelian subgroups*, Quarterly J. Math. **51**(2000), 169-181.

[6] J. F. Carlson and R. Rouquier, *Self-equivalences of stable module categories*, Math. Zeit. **233**(2000), 165-178.

[7] J. F. Carlson and J. Thévenaz, *Torsion endo-trivial modules*, Algebras and Rep. Theory, **3** (2000), 303–335.

[8] J. Carlson and J. Thévenaz, *The classification of torsion endo-trivial modules*, Annals of Math. (to appear).

[9] J. Carlson and J. Thévenaz, *The classification of endo-trivial modules*, Invent. Math. (to appear).

[10] J. Carlson, L. Townsley, L. Valero-Elizondo and M. Zhang, *Cohomology Rings of Finte Groups*, Kluwer, Dordrecht, 2003.

[11] J. Carlson, N. Mazza and D. Nakano *Endotrivial modules for Chevalley groups*, (in preparation).

[12] E. Cartan and S. Eilenberg, *Homological Algebra*, Princeton University Press, Princeton, 1956.

[Ch] L. Chouinard, *Projectivity and relative projectivity over group algebras*, J. Pure Appl. Algebra **7**(1976), 278-302.

[13] E. C. Dade, *Endo-permutation modules over p-groups, I, II*, Ann. Math. **107** (1978), 459–494, **108** (1978), 317–346.

[14] E. Dade, *Une extension de la Théory de Hall et Higman* J. Algebra **20**(1972), 570-609.

[15] L. Evens, "The Cohomology of Groups", Oxford University Press, 1991.

[16] L. Evens and S. Priddy, *The cohomology of the semi-dihedral group*, Conference on algebraic topology in honor of Peter Hilton (St. Johns, Nfld., 1983), 61-72, Contemp. Math. **37**, Amer. Math. Soc., Providence, RI, 1985.

[17] D. Gorenstein, R. Lyons, R. Solomon, *The classification of the finite simple groups*, Number 2, Math. Surveys and Monographs Vol. 40.2, Amer. Math. Soc., Providence, 1996.

[Kr] O. Kroll, *A representation theoretical proof of a theorem of Serre*, preprint (1986).

[18] I. J. Leary, *The mod-p cohomology rings of some p-groups*, Math. Soc. Cam. Phil. Soc. **112** (1992), 63–75.

[Mats] H. Matsumura, *Commutative Ring Theory*, Cambridge University Press, Cambridge, 1986.

[PY] J. Pakianathan and E. Yalcin.

[19] L. Puig, *Notes sur les p-algèbras de Dade*, Preprint, 1988.

[20] L. Puig, *Affirmative answer to a question of Feit*, J. Algebra **131** (1990), 513–526.

[21] D. Quillen, *The spectrum of an equivariant cohomology ring* I, II , Ann. Math. **94**(1971), 549-602.

[22] D. Quillen, *The mod 2 cohomology rings of extra-special 2-groups and the spinor groups*, Math. Ann. **194** (1971), 197–212.

[Se1] J.-P. Serre, *Sur la dimension cohomologique des groupes profinis*, Topology **3**(1965), 413-420.

[23] J. Thévenaz, "*G*-Algebras and Modular Representation Theory", Oxford University Press, 1995.

Department of Mathematics, University of Georgia, Athens, Georgia 30602, USA
E-mail address: jfc@math.uga.edu