Random triangulations of the sphere

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Consider a triangulation of the sphere. This is the same as fixing a triangle and filling it with more triangles (NO double links)

Tutte(1962) : The set \( T \) of triangulations (rooted simplicial 3-polytopes) of the sphere with \( n \) nodes has asymptotically

\[
\frac{3}{16 \sqrt{6} \pi n^5} \left( \frac{256}{27} \right)^{n-3}
\]

elements

Euler: \( n \) nodes, \( 3n - 6 \) links, \( 2n - 4 \) triangles
Dynamics: Given a triangulation $T \in T$, let $P_T(l)$ denote a probability on the links: $\sum_l P_T(l) = 1$

Choose a link with probability $P_T(l)$ and "flip" (if you can)

Repeat above
Tetrahedron is unflippable

Questions:

- "Typical" triangulation (invariant measure on $T$)
- Convergence to this measure

The $P_T(l)$ depend on the nature of the "physical" problem: $\phi^3$ field theory, dual graphs of 2-D foams, "frustrated systems" (glass phase)
Proposition: If all the $P_T(\ell)$ are positive then the Markov chain defined by iterating flips is irreducible and aperiodic

$\Rightarrow$ Unique invariant measure, mixing

- irreducible: every configuration can be reached
- aperiodic: some power of the transition matrix has all elements positive, no "parity"
- One can of course ask a little less than $P_T(\ell) > 0$
Proof of irreducibility (K. Wagner 1936, Negami):
Any configuration can be transformed by flips to a “christmas tree”

It is an induction reducing the number of links at the top
THREE MODELS

Uniform model: Choose a link at random:

\[ P_T(l) = \frac{1}{\#\text{links}} \]

Godrèche, Kostov, Yekutieli (1992). The invariant measure is uniform and the distribution of degrees is exponential when \( n \to \infty \)

- Proof is not trivial, since there are correlations between degrees, and they matter...
Singularity like \((1-x)^{-0.66}\) ?
Magnasco Model: Choose a node, and then a link attached at this node.

This corresponds to

$$P_T(l) = \frac{1}{n} \left( \frac{1}{d_1(l|T)} + \frac{1}{d_2(l|T)} \right)$$

This model has a power law distribution with law about $\text{const} \cdot d^{-4}$

Can one say something about the invariant measure? No. The difficulty is
Theorem: The Markov chain $P(\cdot \mid \cdot)$ is not reversible (when $n \gg 7$). No "detailed balance"

In other words, there is a cycle of 4 flips $T_1 \rightarrow T_2 \rightarrow \cdots \rightarrow T_1$ but the product of the quotients forward/backward is not 1:

$$\prod_{j=1}^{4} \frac{P(T_j \mid T_{j+1})}{P(T_{j+1} \mid T_j)} = \frac{10}{9}$$
Proof is another exercise in flipping
$D(k)/n$

$D(k) = \#\text{nodes of degree } \geq k$, slope is $-3$
Glass Model: work in progress
Hentschel, Ilyin, Makedonska, Procaccia, Schupper
Gas with large blue and small red particles in the plane (on the torus)
with repulsive potential between them
Ground state looks as follows:
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Fix an initial triangulation and color half the nodes blue, the other half, red. Define the following energy for a triangulation $T$:

$$E = \sum_{i=\text{blue}} (\deg_i - 7)^2 + \sum_{i=\text{red}} (\deg_i - 5)^2$$

Monte Carlo: Choose the links uniformly, and flip whenever $E_{\text{after}} \leq E_{\text{before}}$, or flip with probability $\exp(-\beta(E_{\text{after}} - E_{\text{before}}))$, $\beta > 0$.

The process is again aperiodic and irreducible, and satisfies detailed balance. It has invariant density proportional to $\exp(-\beta E_T)$

New difficulty: Have to move the colors around
If the degree is a multiple of 18 there is a state with energy only 6 (probably the minimum). For higher energies, I can show that there are at least $O(c^k)$ triangulations which have energy $E \leq k$. For example, $O(c^{n^{1/2}})$ of energy $E \leq n^{1/2}$.

The distance between any two triangulations is known to be $O(n)$ (also for the colored case). But to reach two T’s with about the same energy, without raising the temperature $1/\beta$, I need to make big “detours”. These facts are responsible for the “glassy” behavior of the system.
NB: Energy $\approx$ number of "defects", i.e., how many $d_{\text{blue}} \neq 7$ resp $d_{\text{red}} \neq 5$. It is a gas of defects, which make a random walk in the space of indices of the nodes, with spontaneous creation and pair-wise annihilation.
Energy as function of time

Defects

Time

10^8 10^9 10^10

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Energy as function of time (collapsed)

% Defects

Time

10^8 10^9 10^10 10^11

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