Embeddings of locally finite metric spaces into Banach spaces.

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PART I

Locally finite metric spaces
1 Locally finite metric spaces

- Definition of the different embeddings
Definition (lipschitz embedding)

Let \((M, d)\) and \((N, \delta)\) be two metric spaces and an injective map \(f : M \to N\). We define the distortion of \(f\) to be

\[
\text{dist}(f) = \| f \|_{\text{Lip}} \| f^{-1} \|_{\text{Lip}}
\]

\[
= \sup_{x \neq y \in M} \frac{\delta(f(x), f(y))}{d(x, y)} \cdot \sup_{x \neq y \in M} \frac{d(x, y)}{\delta(f(x), f(y))}.
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- If \(\text{dist}(f)\) is finite, we say that \(f\) is a lipschitz embedding, or simply an embedding of \(M\) into \(N\).
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- If \(\text{dist}(f)\) is finite, we say that \(f\) is a lipschitz embedding, or simply an embedding of \(M\) into \(N\).

- And if there exists an embedding \(f\) from \(M\) into \(N\), with \(\text{dist}(f) \leq C\), we use the notation \(M \overset{C-\text{lip}}{\hookrightarrow} N\).
Definition (coarse, uniform and strong uniform embeddings)

- Let \((M, d)\) and \((N, \delta)\) be two metric spaces. Suppose \(f : M \to N\) is any map and let

\[
\varphi_f(t) = \inf\{d(f(x), f(y)) : d(x, y) \geq t\} \quad t > 0
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and

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so that

\[
\varphi_f(d(x, y)) \leq d(f(x), f(y)) \leq \omega_f(d(x, y)) \quad \forall x, y \in M.
\]
Then we say that $f$ is a coarse embedding and $M$ coarsely embeds into $N$ if $\omega_f(t) < \infty$ for all $t$ and $\lim_{t \to \infty} \varphi_f(t) = \infty$. We shall refer to $f$ as a strong uniform embedding if it is both a coarse embedding and a uniform embedding.
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On the other hand $f$ is a uniform embedding and $M$ uniformly embeds into $N$ if $\varphi_f(t) > 0$ for all $t > 0$ and $\lim_{t \to 0} \omega_f(t) = 0$. 
Then we say that \( f \) is a coarse embedding and \( M \) coarsely embeds into \( N \) if \( \omega_f(t) < \infty \) for all \( t \) and \( \lim_{t \to \infty} \varphi_f(t) = \infty \).

On the other hand \( f \) is a uniform embedding and \( M \) uniformly embeds into \( N \) if \( \varphi_f(t) > 0 \) for all \( t > 0 \) and \( \lim_{t \to 0} \omega_f(t) = 0 \).

We shall refer to \( f \) as a strong uniform embedding if it is both a coarse embedding and a uniform embedding.
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  ■ Definition of the different embeddings
  ■ Questions and partial answers
Question I

Let $X$ be a Banach space. Do we have: $(\forall \ M \text{ separable metric space}, \ M \xrightarrow{\text{lip}} X) \Rightarrow (c_0 \xrightarrow{\sim} X)$ or equivalently $(c_0 \xrightarrow{\text{lip}} X) \Rightarrow (c_0 \xrightarrow{\sim} X)$. 

[N.J. Kalton]('04) There exists a Banach space $X$ s.t. $c_0$ strongly uniformly embeds into $X$ but $c_0$ does not isomorphically embeds into $X$. 

[N.J. Kalton]('06) If $c_0$ coarsely or uniformly embeds into a Banach space $X$, then there exists $n \in \mathbb{N}$ s.t. $X(n)$ is non-separable.
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If $c_0$ coarsely or uniformly embeds into a Banach space $X$, then there exists $n \in \mathbb{N}$ s.t. $X^{(n)}$ is non-separable.
Question II

*Which metric spaces can be coarsely embedded into a super-reflexive or reflexive Banach space.*
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**Definition**

A metric space $M$ is locally finite if any ball of $M$ with finite radius is finite. If moreover, there is a function $C : (0, +\infty) \to \mathbb{N}$ such that any ball of radius $r$ contains at most $C(r)$ points, we say that $M$ has a bounded geometry.
Let $M$ be a metric space with bounded geometry. There exists a sequence of positive real numbers $\{p_n\}$ and a coarse embedding of $M$ into the $\ell^2$ – direct sum $\bigoplus \ell^{p_n}(\mathbb{N})$. 

[N. Brown, E. Guentner](’05)
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If $M$ is a locally finite metric space then $M$ strongly uniformly embeds into a reflexive Banach space.
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Definition

Let $X$ and $Y$ be two Banach spaces. If $X$ and $Y$ are linearly isomorphic, i.e., there exists a one-to-one and onto linear application, the Banach-Mazur distance between $X$ and $Y$, denoted by $d_{BM}(X, Y)$, is the infimum of $\|T\| \|T^{-1}\|$, over all linear isomorphisms $T$ from $X$ onto $Y$. 
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**Definition**

For $p \in [1, \infty]$ and $n \in \mathbb{N}$, $\ell^n_p$ denotes the space $\mathbb{R}^n$ equipped with the $\ell_p$ norm. We say that a Banach space $X$ uniformly contains the $\ell^n_p$'s if there is a constant $C \geq 1$ such that for every integer $n$, $X$ admits an $n$-dimensional subspace $Y$ so that $d_{BM}(\ell^n_p, Y) \leq C$. 
Theorem (F.B., G. Lancien ’06)

There exists a universal constant $C > 1$ such that for every Banach space $X$ uniformly containing the $\ell^n_\infty$’s and every locally finite metric space $(M, d)$: $M \overset{C}{\hookrightarrow} X$. 
PART II

The hyperbolic tree
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II The hyperbolic tree
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II The hyperbolic tree
  ■ Notation
Denote $\Omega_0 = \{\emptyset\}$, the root of the tree.
Let $\Omega_n = \{-1, 1\}^n$, $T_n = \bigcup_{i=0}^n \Omega_i$ and $T = \bigcup_{n=0}^{\infty} T_n$. 
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Let $\Omega_n = \{-1, 1\}^n$, $T_n = \bigcup_{i=0}^n \Omega_i$ and $T = \bigcup_{n=0}^\infty T_n$.
For $\varepsilon, \varepsilon' \in T$, we note $\varepsilon \leq \varepsilon'$ if $\varepsilon'$ is an extension of $\varepsilon$. 
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Denote $|\varepsilon|$ the length of $\varepsilon$; i.e. the numbers of nodes (or coordinates) of $\varepsilon$. 
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We define the hyperbolic distance between $\varepsilon$ and $\varepsilon'$ by $\rho(\varepsilon, \varepsilon') = |\varepsilon| + |\varepsilon'| - 2|\delta|$, where $\delta$ is the greatest common ancestor of $\varepsilon$ and $\varepsilon'$. 
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$T$ embeds isometrically into $\ell_1(\mathbb{N})$ in a trivial way. Actually, let $(e_{\varepsilon})_{\varepsilon \in T}$ be the canonical basis of $\ell_1(T)$ ($T$ is countable), then the embedding is given by $\varepsilon \mapsto \sum_{s \leq \varepsilon} e_s$. 
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[Bourgain] (’86)

A Banach space $X$ is not super-reflexive if and only if the finite trees $T_n$ uniformly embed into $X$ (i.e with embedding constants independent of $n$).
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Theorem (F.B. ’06)

Let $X$ be a non super-reflexive Banach space, then $(T, \rho)$ embeds into $X$. 