Generalized complex geometry minicourse, Poisson 2008

Problem sheet (Instructor: Marco Gualtieri, mgualt@math.utoronto.ca)

Exercise 1. Let \( \omega \in C^\infty(\bigwedge^2 T^*) \) be nondegenerate, so that the map \( \omega : T \longrightarrow T^* \) defined by

\[
\omega : X \mapsto i_X \omega
\]

is invertible. Show that this is only possible if \( \dim T = 2n \) for some integer \( n \).

Then \( \det \omega : \det T \longrightarrow \det T^* \), or in other words

\[
\det \omega \in \det T^* \otimes \det T^*.
\]

Show that \( \det \omega = (\text{Pf} \ \omega)^2 \), where

\[
\text{Pf} \ \omega = \frac{1}{n!} \omega^n.
\]

Construct the operator \( \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \) on \( T \oplus T^* \) and verify that it is an almost generalized complex structure. What is the \(+i\)-eigenbundle of this operator?

Under what condition is the graph \( \Gamma_{i\omega} = \{ X+i\omega(X) : X \in T \otimes \mathbb{C} \} \) integrable with respect to the Courant bracket?

What is the pure spinor line \( K \subset \bigwedge \cdot \bigwedge T^* \otimes \mathbb{C} \) annihilated by \( \Gamma_{i\omega} \)?

Exercise 2. Let \( J \in \text{End}(T) \) be an almost complex structure, and let \( N_J = [J, J] \) be its Nijenhuis tensor, defined alternatively by

\[
[J, J](X, Y) = [X, Y] - [JX, JY] + J([JX, Y] + [X, JY]).
\]

Defining \( \partial : \Omega^{p,q}(M) \longrightarrow \Omega^{p+1,q}(M) \) by \( \partial = \pi_{p+1,q} d \) and \( \overline{\partial} \) its complex conjugate, then

\[
d = \partial + \overline{\partial} + d_N;
\]

determine the operator \( d_N \) and its decomposition into \((p, q)\) types.

Construct the operator \( \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix} \) on \( T \oplus T^* \) and verify that it is an almost generalized complex structure. What is the \(+i\)-eigenbundle?

Under what condition is the \(+i\)-eigenbundle \( L \) Courant integrable? What is the pure spinor line it annihilates?

Exercise 3. Consider the complex differential form \( \rho = w + dw \wedge dz \), for \((w, z)\) standard coordinates on \( \mathbb{C}^2 \). Show that \( \rho \) is a pure spinor defining a generalized complex structure on \( \mathbb{C}^2 \). What is the type of the structure defined by \( \rho \)? Write the induced generalized complex structure as an operator on \( T \oplus T^* \). What is the real Poisson structure associated to \( \rho \)?
Exercise 4. For a real vector space $V$, let $C_+ \subset V \oplus V^*$ be a maximal positive-definite subspace, and let $C_-$ be its orthogonal complement with respect to the canonical bilinear form $\langle \cdot, \cdot \rangle$ on $V \oplus V^*$. Show that $C_+$ must be the graph of $g + b$ for $g \in S^2V^*$ and $b \in \wedge^2V^*$. Let $G = 1|_{C_+} - 1|_{C_-}$ be the associated generalized metric, so that $\langle G \cdot, \cdot \rangle$ defines a positive-definite metric on $V \oplus V^*$.

- Show that the restriction of $G$ to $V \subset V \oplus V^*$ is

$$g^b = g - bg^{-1}b.$$

Show explicitly that $g^b$ is indeed positive-definite. Also, show that its volume form is given by

$$vol_{g^b} = \det(g - bg^{-1}b)^{1/2} = \det(g + b) \det g^{-1/2}.$$

- Let $(e_i)$ be an oriented g-orthonormal basis for $V$. Show that $(a_i = e_i + (g + b)(e_i))$ form an oriented orthonormal basis for $C_+$. Hence $\ast = a_1 \cdots a_n$ is a generalized Hodge star. Show that $\ast \in \operatorname{Pin}(V \oplus V^*)$ covers $-G \in \operatorname{O}(V \oplus V^*)$.

- Show explicitly that the Mukai pairing $(\ast 1, 1) = \det(g + b) \det g^{-1/2} = vol_{g^b}$.

- Show that $vol_{g^b}/vol_g = ||e_b||_g^2$ (Hint: determine the relationship between $\ast_g$ and $\ast_{g^b}$.)

Exercise 5. Show that the derived bracket expression $[(a,b)]_H \cdot \varphi = [[d_H,a],b]\cdot \varphi$ for the twisted Courant bracket (where $d_H = d + H \wedge \cdot$) agrees with that obtained from the axioms of an exact Courant algebroid, i.e.

$$[X + \xi,Y + \eta]_H = [X,Y] + L_X\eta - iy d\xi + iy i_X H.$$

Exercise 6. Let $[\cdot,\cdot]$ be the derived bracket on $\mathcal{C}^\infty(T \oplus T^*)$ of the operator $d_H = d + H \wedge \cdot$ but do not assume that $dH = 0$. Prove that

$$[[a,b],c] = [a,[b,c]] - [b,[a,c]] + i_{\pi c} i_{\pi b} i_{\pi a} dH.$$

Exercise 7. Let $\pi : T^* \to T$ be a Poisson structure with associated Poisson bracket $\{\cdot,\cdot\}$. Show that $T^*$ inherits a natural Lie algebroid structure, where $\pi$ is the anchor map and

$$[df,dg] = d\{f,g\}.$$

Exercise 8. Let $\beta, \beta'$ be gauge equivalent Poisson structures, i.e.

$$\beta' = \beta(1 + B\beta)^{-1}$$

for $B \in \Omega^2(M,\mathbb{R})$ and such that the inverse above exists. Verify that $\beta'$ is indeed Poisson, and show that there is a canonical isomorphism between the $\beta$ and $\beta'$ Poisson cohomology groups.
Exercise 9. Let \( L \subset T \oplus T^* \) be an \( H \)-twisted Dirac structure on \( N \) and let \( M \subset N \) be a leaf of the generalized distribution \( \Delta = \pi_T(L) \) on \( N \). Then \( L \) determines canonically a 2-form \( \epsilon \in \Omega^2(M) \). Show using the integrability of \( L \) that \( de = f^*H \), where \( f : M \to N \) is the inclusion map.

Exercise 10. Let \( L \) be the complex Dirac structure associated to a generalized complex structure \( J \), and let \( K_L \) be the complex pure spinor line defining \( L \). Use the Mukai pairing to demonstrate that \( 2c_1(K_L) = c_1^+ + c_1^- \) where \( c_1^\pm \) are the first Chern classes of the \( U(n,n) \) structure defined by \( J \). Explain why \( c_1^+ + c_1^- \) must be even a priori.

Exercise 11. Let \( J \) be a generalized complex structure on the exact Courant algebroid \( E \) such that \( JT^* = T^* \). Write the decomposition of \( J \) given a general (non-complex) splitting \( s : T \to E \). Hint: determine the difference between the splittings \( s \) and \( -JsJ \), where \( J \) is the induced complex structure on \( E/T^* \). How does this compare to the expression of \( J \) in a complex splitting?

Exercise 12. Let \( J \) be an almost generalized complex structure. Show that
\[
N_J(x,y) = [Jx,Jy] - J[Jx,y] - J[x,Jy] - [x,y]
\]
is tensorial, and vanishes if and only if \( J \) is integrable.

Exercise 13. Let \( J \) be a generalized complex structure. Show that \( e^{\theta J}(T^*) \) is a Dirac structure for all \( \theta \).

Exercise 14. Let \( (g,I,J) \) define a hyperKähler structure, so that \( JT^* = T^* \). Let \( \omega_I, \omega_J, \omega_K \) be the associated symplectic forms. Verify that for \( a,b,c \) real and \( a^2 + b^2 + c^2 = 1 \),
\[
J(a,b,c) = aJ_I + bJ_J + cJ_K
\]
squares to \(-1\), and is an orthogonal endomorphism of \( T \oplus T^* \). Also prove that for \( a \neq 0 \), \( J(a,b,c) \) is a B-field transform of a symplectic structure. Conclude that \( J(a,b,c) \) is an integrable generalized complex structure for all points on the sphere.

Exercise 15. Give examples of 0- and 2-branes in Example 3. Are there any 4-branes? What about branes of odd dimension?

Exercise 16. Are there 2-branes in the \( \beta \)-deformed \( \mathbb{C}P^2 \) which are not complex curves in \( \mathbb{C}P^2 \)? Are there any 4-branes?

Exercise 17. Let \( J \) be an even generalized complex structure. Then the canonical pure spinor line is a sub-bundle
\[
K \subset \Lambda^*T^* \otimes \mathbb{C}.
\]
the projection \( s : K \to \Lambda^0T^* \otimes \mathbb{C} \) defines a section \( s \in C^\infty(K^*) \). Show that \( \overline{\partial}s = 0 \), where \( \overline{\partial} \) is the generalized Dolbeault operator induced by the generalized complex structure.

Exercise 18. Give an example of a Generalized Kähler structure where both \( J_A, J_B \) are of symplectic type.