Generalized reduction and supergeometry

Henrique Bursztyn (IMPA)
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Reduction of generalized complex structures

Supergeometric viewpoint
(Joint with Cattaneo, Mehta, Zambon)
Usual Scenario - Hamiltonian actions
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Geometrical structure: $\omega \in \Omega^2(M)$, symplectic form
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Ingredients for reduction
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Action: $G$-action on $M$, $\psi : \mathfrak{g} \rightarrow \Gamma(TM)$
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Geometrical structure: \( \omega \in \Omega^2(M) \), symplectic form

Ingredients for reduction

Action: \( G \)-action on \( M \), \( \psi : \mathfrak{g} \rightarrow \Gamma(TM) \)

Equivariant map: \( \mu : M \rightarrow \mathfrak{g}^* \)
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**Action:** $G$-action on $M$, $\psi : \mathfrak{g} \rightarrow \Gamma(TM)$

**Equivariant map:** $\mu : M \rightarrow \mathfrak{g}^*$

**Moment map condition:** $i_{\psi(u)}\omega = d\langle \mu, u \rangle$
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Marsden-Weinstein reduction:

$M_{red} = \mu^{-1}(0)/G$ inherits symplectic structure $\omega_{red}$ ...
Generalized Scenario (outline of the talk)
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1. **Geometrical structure**: $\mathcal{J}$, generalized complex structure
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2. **(Lifted) Action**: $\tilde{\psi} : g \rightarrow \Gamma(TM \oplus T^*M)$
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Ingredients for reduction

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3. **Equivariant map**: \( \mu : M \rightarrow \mathfrak{h}^* \), \( \mathfrak{h} \) is \( G \)-module
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4. **Moment map condition:** compatibility involving $\mathcal{J}, \tilde{\psi}, \mu$ ...
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6. Generalized reduction is (super) Marsden-Weinstein reduction
1. **Generalized complex structures** (Hitchin ’03, Gualtieri ’04)
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Courant bracket (“H-twisted”, $H \in \Omega^3_{cl}(M)$):

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\llbracket (X, \xi), (Y, \eta) \rrbracket := ([X, Y], \mathcal{L}_X \eta - i_Y d\xi + i_Y i_X H).
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**Generalized complex structure** on $M$:

$\mathcal{J} : \mathbb{T}M \rightarrow \mathbb{T}M$, $\mathcal{J}^2 = -1$, such that $\mathcal{J} \in O(\mathbb{T}M)$ and

$$[\Gamma(L), \Gamma(L)] \subset \Gamma(L), \quad (\text{integrability condition})$$
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$$\llbracket \Gamma(L), \Gamma(L) \rrbracket \subset \Gamma(L), \quad \text{(integrability condition)}$$

where $L = +i$-eigenbundle of $\mathcal{J}$. 
Example: $J : TM \to TM$, $J^2 = -1$,

$$\mathcal{J}_J = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}, \quad L_J = T_{10} \oplus T_{01}^*$$

integrability $\iff [T_{10}, T_{10}] \subset T_{10}$
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Example: $\omega : TM \to T^*M$, nondegenerate 2-form, 

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}, \quad L_\omega = \text{graph}(-i\omega)$$

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Many other examples ....

- Holomorphic Poisson structures (type change...)
- dim 4 (not complex, not symplectic)
2. (Lifted) Actions

“New” tangent bundle: $$(TM, [\cdot, \cdot]) \mapsto (TM, \langle \cdot, \cdot \rangle, [\cdot, \cdot], H),$$
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Lifted Action:

\[
\begin{array}{c}
g \\
\xrightarrow{\tilde{\psi}} \\
\xrightarrow{\psi} \\
\Gamma(TM)
\end{array}
\quad \xrightarrow{\psi} \quad \\
\tilde{\psi}(u) = (\psi(u), \sigma(u))
\]

such that:
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such that:

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\Diamond \tilde{\psi} : g \rightarrow \Gamma(TM) \text{ is bracket preserving.}
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such that:

\[\diamond \tilde{\psi} : \mathfrak{g} \to \Gamma(TM) \text{ is bracket preserving.}\]

\[\diamond \text{Im}(\tilde{\psi}) \subset TM \text{ is isotropic,}\]
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**Lifted } G\text{-Action:**} G\text{-action on } TM\text{ integrating } u \mapsto [\tilde{\psi}(u), \cdot].
2. (Lifted) Actions

“New” tangent bundle: $(TM, [~, ~]) \leadsto (\mathbb{T}M, \langle *, * \rangle, [~, ~], H)$,

Lifted Action:
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\end{array}
\]
\[
\tilde{\psi}(u) = (\psi(u), \sigma(u))
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such that:

• $\tilde{\psi} : \mathfrak{g} \to \Gamma(TM)$ is bracket preserving.

• $\text{Im}(\tilde{\psi}) \subset TM$ is isotropic,

Lifted $G$-Action: $G$-action on $TM$ integrating $u \mapsto [\tilde{\psi}(u), ~]$.

How to find $\sigma : \mathfrak{g} \to \Omega^1(M)$ defining lift $\tilde{\psi} = (\psi, \sigma)$?
Remark: If $\sigma = 0$, 

$$
\psi([u, v]) = [\psi(u), \psi(v)] - i_{\psi(u) \wedge \psi(v)} H.
$$
Remark: If $\sigma = 0$,

$$\psi([u, v]) = [\psi(u), \psi(v)] - i_{\psi(u) \wedge \psi(v)}H.$$  

Example: $G$-action on $M$, consider Cartan complex:

$$\Omega^k_G(M) = \bigoplus_{2p+q=k} (S^p(g^*) \otimes \Omega^q(M))^G,$$

$$(d_G\Phi)(u) = d(\Phi(u)) - i_{\psi(u)}(\Phi(u)).$$

If $\sigma : g \to \Omega^1(M)$,

$$\sigma + H \in \Omega^3_G(M), \quad d_G(\sigma + H) = 0,$$

then

- $\diamond \sigma$ defines a lifted action,
- $\diamond$ Induced action on $\mathbb{T}M$ is the canonical one.
3. Equivariant (moment) maps

Take $G$-module $\mathfrak{h}$, consider equivariant map

$$\mu : M \to \mathfrak{h}^*$$

(Important to allow $\mathfrak{h} \neq \mathfrak{g} \ldots$)
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**Reduction data:** $(\tilde{\psi}, \mathfrak{h}, \mu)$, where

- Lifted action $\tilde{\psi} : \mathfrak{g} \to \Gamma(TM \oplus T^*M)$,
- Equivariant $\mu : M \to \mathfrak{h}^*$. 
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Regularity: 0 is regular value, $G$-action on $\mu^{-1}(0)$ is free/proper.
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**Regularity:** $0$ is regular value, $G$-action on $\mu^{-1}(0)$ is free/proper.

**Reduced space:** $M_{red} := \mu^{-1}(0)/G$.

**Remark:**

$$
\mathfrak{h} = 0, \quad M_{red} = \frac{M}{G}, \quad \text{and} \quad \mathfrak{g} = 0, \quad M_{red} = \mu^{-1}(0).
$$
4. Moment map condition

- Manifold $M$, $H \in \Omega^3_{cl}(M)$,
- Reduction data $(\tilde{\psi}, \mathfrak{h}, \mu)$,
- Generalized complex str. $\mathcal{J}$ on $M$ ($G$-invariant).

Consider

$$K := \{ \tilde{\psi}(u) + d\langle \mu, v \rangle \mid u \in \mathfrak{g}, v \in \mathfrak{h} \} \subset TM \oplus T^*M.$$
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- Manifold \( M \), \( H \in \Omega^3_{cl}(M) \),
- Reduction data \((\tilde{\psi}, \mathfrak{h}, \mu)\),
- Generalized complex str. \( \mathcal{J} \) on \( M \) (\( G \)-invariant).

Consider

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**Moment map condition:**

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\mathcal{J}K = K, \quad \text{(over} \ \mu^{-1}(0))\).
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Example: Hamiltonian action $G$ on $(M, \omega)$,

$$\mu : M \to \mathfrak{g}^*, \ i_{\psi(u)}\omega = d\langle \mu, u \rangle$$
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Reduction data: $(\psi, \mathfrak{h} = \mathfrak{g}, \mu)$.

$$K = \{ (\psi(u), d\langle \mu, v \rangle) \mid u \in \mathfrak{g}, v \in \mathfrak{h} \} = \psi(\mathfrak{g}) \oplus d\langle \mu, \mathfrak{g} \rangle.$$
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Example: Hamiltonian action $G$ on $(M, \omega)$,

$$\mu : M \to g^*, \ i_{\psi(u)}\omega = d\langle \mu, u \rangle$$

Reduction data: $(\psi, h = g, \mu)$.

$$K = \{ (\psi(u), d\langle \mu, v \rangle) \mid u \in g, v \in h \} = \psi(g) \oplus d\langle \mu, g \rangle.$$ 

Moment map condition:

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Example: Holomorphic action $G$ on $(M, J)$, $\psi(\imath u) = J\psi(u)$

Reduction data: $(\psi, h = \{0\}, \mu = 0)$.

$$K = \{ \psi(u) \mid u \in g \} = \psi(g) \subset TM.$$
**Example:** Hamiltonian action $G$ on $(M, \omega)$,

$$\mu : M \to \mathfrak{g}^*, \ i_{\psi(u)} \omega = d\langle \mu, u \rangle$$

Reduction data: $(\psi, \mathfrak{h} = \mathfrak{g}, \mu)$.

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Moment map condition:

$$\begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix} \begin{pmatrix} \psi(u) \\ 0 \end{pmatrix} = \begin{pmatrix} \psi(iu) \\ 0 \end{pmatrix} \in K$$
5. Reduction

Set-up:

◇ Reduction data: \((\tilde{\psi}, \mathfrak{h}, \mu)\)

◇ \(G\)-invariant GCS: \(\mathcal{J}: T\mathcal{M} \rightarrow T\mathcal{M}\).

Goal is to reduce \(\mathcal{J}\) to \(\mathcal{M}_{\text{red}} = \mu^{-1}(0)/G\).
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Set-up:

- **Reduction data:** $(\tilde{\psi}, \mathfrak{h}, \mu)$
- **$G$-invariant GCS:** $\mathcal{J} : T\mathcal{M} \to T\mathcal{M}$.

Goal is to reduce $\mathcal{J}$ to $M_{red} = \mu^{-1}(0)/G$.

Reduction divided in two steps:

(a) **Reduction of Courant structure** $(T\mathcal{M}, \langle \cdot, \cdot \rangle, [\cdot, \cdot])$

(Which $H_{red}$ to use on $M_{red}$?)

(b) **Reduction of** $\mathcal{J}$
(a) Reduction of the Courant bracket
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Let $K = \{ \tilde{\psi}(u) + d\langle \mu, v \rangle \mid u \in \mathfrak{g}, v \in \mathfrak{h} \}$.

Regularity $\implies K|_{\mu^{-1}(0)}$ is vector bundle, isotropic.
(a) Reduction of the Courant bracket

Let \( K = \{ \tilde{\psi}(u) + d\langle \mu, v \rangle \mid u \in g, v \in h \} \).

Regularity \( \implies K|_{\mu^{-1}(0)} \) is vector bundle, isotropic.

**Theorem** [BCG’07]

\( K|_{\mu^{-1}(0)} \) is equivariant \( G \)-bundle, and the quotient

\[
E_{\text{red}} := \frac{K^\perp|_{\mu^{-1}(0)}}{K|_{\mu^{-1}(0)}} \bigg/ G \longrightarrow M_{\text{red}}
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is exact Courant algebroid \((M_{\text{red}} = \mu^{-1}(0)/G)\).
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This means: \( E_{\text{red}} \cong T M_{\text{red}} \oplus T^* M_{\text{red}} \), but noncanonically!

Description of Ševera class \([H^{\text{red}}]\) of \( E_{\text{red}} \).
Example:
Usual action $\psi : \mathfrak{g} \rightarrow \Gamma(TM) \ (H = 0, \ \sigma = 0), \ \mu : M \rightarrow \mathfrak{h}^*$,

$$E_{red} = T(M_{red}) \oplus T^*(M_{red})$$

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Example: $M = S^3 \times S^1$, $H = 0$. 
Example:
Usual action $\psi : g \rightarrow \Gamma(TM)$ $(H = 0, \sigma = 0)$, $\mu : M \rightarrow \mathfrak{h}^*$,

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Example: $M = S^3 \times S^1$, $H = 0$.

$S^1$ action on $M$: on $S^3$, generates Hopf bundle $S^3 \rightarrow S^2$,
on $S^1$, trivial action.
Example:
Usual action $\psi : \mathfrak{g} \to \Gamma(TM)$ ($H = 0$, $\sigma = 0$), $\mu : M \to \mathfrak{h}^*$,
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$S^1$ action on $M$: on $S^3$, generates Hopf bundle $S^3 \to S^2$,
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$\xi \in \Omega^1(S^1)$ invariant volume form.

Lifted action: $\tilde{\psi} : \mathbb{R} \to \Gamma(TM)$, $1 \mapsto \psi(1) + \xi$.

Reduced class: $H_{\text{red}} = F \wedge \xi$, $F$ is Chern class of Hopf fibration.
(b) Reduction of generalized complex structures

How to transport $\mathcal{J}$ to $E_{\text{red}}$?
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How to transport $\mathcal{J}$ to $E_{\text{red}}$?

Let $L \subseteq TM \otimes \mathbb{C}$ be $+i$-eigenbundle ($G$-inv Dirac structure).

Consider $L_{\text{red}} := \frac{K_C^\perp \cap L + K_C}{K_C} \bigg|_{\mu^{-1}(0)}/G$ in $E_{\text{red}}$. 
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Let $L \subseteq TM \otimes \mathbb{C}$ be $+i$-eigenbundle (G-inv Dirac structure).

Consider $L_{red} := \frac{K_{\mathbb{C}}^\perp \cap L + K_{\mathbb{C}}}{K_{\mathbb{C}}} \mid_{\mu^{-1}(0)} / G$ in $E_{red}$.

**Theorem**[BCG’07] If

$$\mathcal{J} K = K \quad \text{over } \mu^{-1}(0),$$

then $L_{red}$ defines a reduced GCS, $\mathcal{J}^{red} : E_{C}^{red} \rightarrow E_{\mathbb{C}}^{red}$. 
(b) Reduction of generalized complex structures

How to transport $J$ to $E_{red}$?

Let $L \subseteq \mathbb{T}M \otimes \mathbb{C}$ be $+i$-eigenbundle ($G$-inv Dirac structure).

Consider $L_{red} := \frac{K_C^+ \cap L + K_C}{K_C} \mid_{\mu^{-1}(0)} / G$ in $E_{red}$.

**Theorem** [BCG’07] If

$$JK = K \quad \text{over } \mu^{-1}(0),$$

then $L_{red}$ defines a reduced GCS, $J_{red} : E_{C}^{red} \to E_{C}^{red}$.

This means: $L_{red}$ is smooth Dirac structure and $L \cap \overline{L} = \{0\}$. 
Examples:

- Hamiltonian reduction: \((\psi, g, \mu); \ J_\omega^{\text{red}} = J_{\omega^{\text{red}}}\).
- Holomorphic quotients: \((\psi, \mathfrak{h} = \{0\}, \mu = 0); \ J_f^{\text{red}} = J_{J^{\text{red}}}\).
Examples:

- **Hamiltonian reduction**: \((\psi, g, \mu); \quad J^\text{red}_\omega = J^\omega_\text{red}\).  
- **Holomorphic quotients**: \((\psi, h = \{0\}, \mu = 0); \quad J^\text{red}_f = J^f_\text{red}\).  
- **Generalized reduction Lin-Tolman/Hu**: \((\tilde{\psi}, h, \mu), \text{ where}\)
  
  - \(\tilde{\psi}\) defined by closed equiv extension \(\sigma\),
  - \(h = g, \mu : M \to g^*\),
  - Compatibility: \(J(d\langle \mu, u \rangle) = \tilde{\psi}(u) (\implies J K = K)\).
Examples:

- Hamiltonian reduction: \((\psi, g, \mu)\); \(J_\omega^{red} = J_{\omega red}\).
- Holomorphic quotients: \((\psi, h = \{0\}, \mu = 0)\); \(J_f^{red} = J_{J_{red}}\).
- Generalized reduction Lin-Tolman/Hu: \((\tilde{\psi}, h, \mu)\), where
  - \(\tilde{\psi}\) defined by closed equiv extension \(\sigma\),
  - \(h = g\), \(\mu : M \to g^*\),
  - Compatibility: \(J(d\langle \mu, u \rangle) = \tilde{\psi}(u) (\implies JK = K)\).
- Symplectic type \(\leadsto\) Complex type:
  - \(M = T^2 \times T^2\), \(T^2\)-bundle over \(T^2\), lagrangian fibres.
  - \(\tilde{\psi}(e_1) = \psi(e_1) + i_{\psi(e_2)}\omega\), \(\tilde{\psi}(e_2) = -\psi(e_2) + i_{\psi(e_1)}\omega\); \(h = \{0\}\).
  - \(J_\omega K = K\) and \(J_\omega^{red}\) on \(M_{\text{red}} = T^2\) is complex.
Other examples from generalized Kähler reduction...
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\( \mathcal{G} : \mathbb{T}M \to \mathbb{T}M \) generalized metric (\( \mathcal{G} \) orthog, self-adjoint, positive).

Compatibility: \( \mathcal{J} \mathcal{G} = \mathcal{G} \mathcal{J} \).

**Theorem** [BCG’08] If

\[
\mathcal{J} K^\mathcal{G} = K^\mathcal{G} \quad \text{over } \mu^{-1}(0),
\]

then \( \mathcal{J} \) and \( \mathcal{G} \) reduce to \( E^{\text{red}} \) and commute.

Here \( K^\mathcal{G} = \mathcal{G} K^\perp \cap K^\perp \).
Other examples from generalized Kähler reduction...

\[ G : \mathbb{T}M \to \mathbb{T}M \] generalized metric (\( G \) orthog, self-adjoint, positive).
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then \( \mathcal{J} \) and \( G \) reduce to \( E^{\text{red}} \) and commute.

Here \( K^G = G K^\perp \cap K^\perp \).

- Examples include Kähler/Hyper-Kähler reduction
- Application include construction of bihermitian structures (e.g. \( \mathbb{C}P^2 \), reducing deformed GKS on \( \mathbb{C}^3 \))
Classical/Generalized reductions
**Classical/Generalized reductions**

<table>
<thead>
<tr>
<th>Symplectic: $\omega \in \Omega^2(M)$</th>
<th>Generalized complex: $\mathcal{J}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Action:</strong> $\psi : \mathfrak{g} \to \Gamma(TM)$</td>
<td><strong>Lifted action:</strong> $\tilde{\psi} : \mathfrak{g} \to \Gamma(TM)$</td>
</tr>
<tr>
<td><strong>Moment map:</strong> $\mu : M \to \mathfrak{g}^*$</td>
<td><strong>Moment map:</strong> $\mu : M \to \mathfrak{h}^*$</td>
</tr>
<tr>
<td>$i_{\psi(u)} = d\langle \mu, u \rangle$</td>
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<td><strong>Reduction:</strong> $\omega_{\text{red}}$ on $\mu^{-1}(0)/G$</td>
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</table>
6. The supergeometric viewpoint

Supergeometric approach to Courant algebroids:
(Vaintrob, Weinstein, Ševera, Roytenberg)

Pseudo-euclidean vector bundle $\iff$ degree 2 symplectic $N$-manifolds

$$E \rightarrow M, \langle \cdot, \cdot \rangle \quad (\mathcal{M}, \Omega)$$

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\]

Courant structure \([\cdot, \cdot], \rho_E\) \( \iff \) Degree 3 function \( \Theta, \{ \Theta, \Theta \} = 0 \)

\[
C^\infty(M) \xrightarrow{\rho^*d} \Gamma(E) \xrightarrow{[e, \cdot]} \text{sym}(E, \langle \cdot, \cdot \rangle) \quad \quad [e_1, e_2] = \{e_1, \Theta\}, e_2, \rho_E(e).f = \{e, \Theta\}, f \]
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\end{align*}
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Gen. Complex structure \(\mathcal{J}\) \(\leftrightarrow\) Degree 2 function \(\mathcal{J}\),
\[
\{\{\Theta, \mathcal{J}\}, \mathcal{J}\} = -\Theta
\]

(J. Grabowski)
Hamiltonian actions on $(M, \Omega)$, compatible with $\Theta$

**Symmetries:** $\tilde{g}$ graded Lie algebra, degrees $-2, -1, 0$. 
Hamiltonian actions on \((\mathcal{M}, \Omega)\), compatible with \(\Theta\)

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**Reduction:** \(C^\infty(\mathcal{M}_{red}) = N(\mathcal{I})/\mathcal{I}\) inherits \(\Omega_{red}\),

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**Reduction of $\Theta$:** Needs $\{\Theta, \mathcal{I}\} \subset \mathcal{I}$.

**Suffices:** $(\tilde{\mathfrak{g}}, [\cdot, \cdot], \delta)$ is DGLA and $\tilde{\mu}(\delta u) = \{\Theta, \tilde{\mu}(u)\}$. 
Example:

Reduction data: $\tilde{\psi}: \mathfrak{g} \to \Gamma(TM)$, $G$-module $\mathfrak{h}$, $\mu: M \to \mathfrak{h}^*$. 
Example:

Reduction data: $\tilde{\psi}: g \rightarrow \Gamma(TM)$, $G$-module $\mathfrak{h}$, $\mu : M \rightarrow \mathfrak{h}^*$.

GLA: $\tilde{g} = g \oplus a[-1] \oplus h[-2]$, where $a = g \oplus h$.

$[u_1, u_2], \ [u_1, v_2] = u_1.v_2, \ [(u_1, v_1), (u_2, v_2)] = (0, u_1.v_2 + u_2.v_1)$

Differential: $\mathfrak{h} \rightarrow a \rightarrow g$
Example:

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**Differential:** $\mathfrak{h} \to \mathfrak{a} \to \mathfrak{g}$

**Hamiltonian action:** $\tilde{\mu} : \tilde{\mathfrak{g}}[2] \to C^\infty(M)$,

- $\mathfrak{h} \to C^\infty(M)$, $v \mapsto \langle \mu, v \rangle$,
- $\mathfrak{a} \to \Gamma(TM)$, $(u, v) \mapsto \tilde{\psi}(u) + d\langle \mu, v \rangle$,
- $\mathfrak{g} \to C^\infty_2(M)$, $u \mapsto \{\Theta, \tilde{\psi}(u)\}$ \quad \(X_{\{\Theta, \tilde{\psi}(u)\}} = \psi(u)\)
Example:

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$$\mathfrak{g} \to \mathcal{C}_2^\infty(M), \ u \mapsto \{\Theta, \tilde{\psi}(u)\} \quad (X_{\{\Theta, \tilde{\psi}(u)\}} = \psi(u))$$

Reduction of Courant bracket: $(\mathcal{M}_{\text{red}}, \Omega_{\text{red}}, \Theta_{\text{red}})$ corresponds to $E_{\text{red}}$. 
Example:

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- $\mathfrak{g} \to C^\infty_2(\mathcal{M}), u \mapsto \{\Theta, \tilde{\psi}(u)\} \quad (X_{(\Theta, \tilde{\psi}(u))} = \psi(u))$

Reduction of Courant bracket: $(\mathcal{M}_{\text{red}}, \Omega_{\text{red}}, \Theta_{\text{red}})$ corresponds to $E_{\text{red}}$.

Reduction of GCS: $\mathcal{J} K = K \implies \mathcal{J} \in C^\infty_2(\mathcal{M})$ descends

$\{\mathcal{J}, \mathcal{I}\} \subset \mathcal{I}$
Generalized/Super symplectic reductions
**Generalized/Super symplectic reductions**

<table>
<thead>
<tr>
<th>$(TM \oplus T^*M, \langle \cdot, \cdot \rangle, [[\cdot, \cdot]])$</th>
<th>$(\mathcal{M}, \Omega, \Theta)$</th>
</tr>
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<tbody>
<tr>
<td>$\mathcal{J} : TM \to TM$, GCS</td>
<td>$\mathcal{J} \in C^\infty_2(\mathcal{M})$, ${{\Theta, \mathcal{J}}, \mathcal{J}} = -\Theta$</td>
</tr>
<tr>
<td><strong>Reduction data:</strong> $(\tilde{\psi}, \hbar, \mu)$</td>
<td>DGLA $\tilde{\mathfrak{g}}$, $\tilde{\mu} : \tilde{\mathfrak{g}} \to C^\infty(\mathcal{M})$</td>
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