Noncommutative Geometry and Quantization for Singular Foliations

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Let \((M, \mathcal{F})\) a foliated manifold (singular). Consider the problems:

**Quantization**
- What is the meaning of \(\mathcal{F}^*\)?
- Does it have a Lie-Poisson structure as in the regular case?
- If so, how can we quantize it?

**Analysis**
Let \(\mathcal{F} = \langle X_1, \ldots, X_n \rangle\) and \(\Delta = \sum X_i^2\). It’s an essentially self-adjoint operator on \(L^2(\text{leaf})\).
- Where does the resolvent of \(\Delta\) live?
- Where does the index of \(\Delta\) live?

**Geometry**
- Is there a Čech/de Rham cohomology \(H(M/\mathcal{F})\)?
Bi-submersions I

We need to record the holonomies of $\mathcal{F}$. No transversals here!

**Definition**

A bi-submersion is a triple $(U, t, s)$ where $s, t : U \to M$ are submersions and

$$s^{-1}(\mathcal{F}) = t^{-1}(\mathcal{F}) = \Gamma \ker ds + \Gamma \ker dt$$

Every $u \in U$ records a longitudinal local diffeomorphism: Just take a local bisection $V$ at $u$ and put

$$t \circ s^{-1} : s(V) \to t(V)$$

**Proposition**

If $(U_i, t_i, s_i)$ are bi-submersions, $i = 1, 2$ then $u_1$ and $u_2$ record the same diffeo iff $u_2 = \phi(u_1)$ for some $\phi : U_1 \to U_2$ that commutes with $s, t$. We say $U_1$ is adapted to $U_2$. 
Bi-submersions II

e.g. $U_1 \circ U_2 = U_1 \times_{s,t} U_2$ is adapted to some $U$.

**Definition**

An **atlas** of $\mathcal{F}$ is a family $\mathcal{U} = (U_i, s_i, t_i)_{i \in I}$ of bi-submersions such that

- $\bigcup_{i \in I} s_i(U_i) = M$
- Every $U_i \circ U_j$ is adapted to $\mathcal{U}$
- If $(\mathcal{U}, t, s) \in \mathcal{U}$ then $(\mathcal{U}, s, t)$ is adapted to $\mathcal{U}$.

Where do we find such an atlas?

- Every Lie groupoid $G \rightrightarrows M$ defining $\mathcal{F}$ (if it exists) is an atlas.
- If $\mathcal{F}$ is regular, the holonomy groupoid is an atlas.

Both of them record the identity diffeo.
**Theorem**

There exists a minimal atlas $\mathcal{U}$ near the identity for a foliation $\mathcal{F}$.

**Proof**

$I_x = \{ \text{functions in } C^\infty(M) \text{ vanishing at } x \}$. Put $\mathcal{F}_x = \mathcal{F}/I_x\mathcal{F}$. Get

$$0 \to g_x \to F_x \xrightarrow{\text{ev}_x} F_x \to 0$$

- If $\dim(\mathcal{F}_x) = n$ then Frobenius $\Rightarrow \mathcal{F} = \langle X_1, \ldots, X_n \rangle$ near $x$.
- $t(y, \lambda) = \exp(\sum_{i=1}^{n} \lambda_i X_i)(y)$ in nhd $U$ of $(x, 0) \in M \times \mathbb{R}^n$.

$(U, t, s)$ is a bi-submersion, with $s = \text{pr}_1$. **Properties:**

- $U$’s record path holonomies $\Rightarrow$ any bi-subm. adapted to a $U$.
- Holonomy groupoid $\text{Hol}(\mathcal{F}) \twoheadrightarrow M$: quotient of this atlas by the adaptation equivalence relation.
Convolutions algebra

For \((U, t, s) \in U\) put \(\Omega^{1/2}(U) = \Omega^{1/2}(\ker ds \times \ker dt)\). Define

\[
\mathcal{A}_U = \bigoplus_{i \in I} C^\infty_c(U_i; \Omega^{1/2}(U_i))/\sim
\]

and \(Q_U : \Omega^{1/2}(U) \to \mathcal{A}_U\) the quotient map. Here

\[
\Omega^{1/2}(U_1) \ni f_1 \sim f_2 \in \Omega^{1/2}(U_2) \iff f_2 = \phi_!(f_1)
\]

where \(\phi_!\) is integration along the fibers of submersion \(\phi : U_2 \to U_1\) which commutes with source and target.

- **Convolution**: \(Q_{U_1}(f_1) \ast Q_{U_2}(f_2) = Q_{U_1 \circ U_2}(f_1, f_2)\)
- **Involution**: \((Q_U(f))^* = Q_{\bar{U}}(f^*)\), where \(\bar{U} = U, \bar{s} = t, \bar{t} = s\).
- **Left regular representation**: \(\pi_x : \mathcal{A}_U \to \mathcal{L}(L^2(Hol(\mathcal{F}_x)))\)
  (convolution formula plus appropriate \(L^1\)-estimate.)

\[C^*(M, \mathcal{F}) = \text{completion of } \mathcal{A}_U \text{ by } \|f\| = \sup_{x \in M} \|\pi_x(f)\|\].
Quantization I: The cotangent bundle

Fix a foliated manifold \((M, \mathcal{F})\) and consider the path holonomy atlas \(\mathcal{U} = \{(U_i, t_i, s_i)\}_{i \in I}\).

**Definition**

Every \(\xi \in \mathcal{F}_x^*\) with \(x \in M\) defines a linear functional on \(\mathcal{F}\) by

\[
\xi \circ e_x : \mathcal{F} \to \mathcal{F}_x \to \mathbb{R}
\]

Consider the space

\[
\mathcal{F}^* = \bigcup_{x \in M} \mathcal{F}_x^*
\]

endowed with the topology of pointwise convergence on \(\mathcal{F}\). It is a **locally compact space**.

**Proposition**

Locally \(\mathcal{F}^*\) is a closed subspace of a trivial bundle \((\mathbb{R}^n)^* \times M\). We call it the **cotangent bundle** of the foliation \(\mathcal{F}\).
Quantization II: Connes’ tangent groupoid

\( \mathcal{F} \): foliation on \( M \). Define foliation on \( M \times \mathbb{R} \) with leaves:
- \( (x, 0) \) for \( x \in M \);
- \( L \times \lambda \) if \( \lambda \neq 0 \). \( L \) is a leaf of \( \mathcal{F} \) on \( M \).

The module is \( \hat{\mathcal{F}} \subseteq \Gamma(T(M \times \mathbb{R})) \) generated by \( (\lambda X, 0) \) where \( \lambda : M \times \mathbb{R} \to \mathbb{R} \) is the second projection.

**Bi-submersions:** If \( (U, t, s) \) is a bi-submersion of \( \mathcal{F} \) then a bi-submersion of \( \hat{\mathcal{F}} \) is \( (\hat{U}, \hat{t}, \hat{s}) \), where

\[
\hat{U} = TU \times \{0\} \cup U \times \mathbb{R}^*
\]

- Its holonomy groupoid is a field of groupoids \( (\hat{G}_\lambda)_{\lambda \in \mathbb{R}} \).
- For \( \lambda \neq 0 \), \( \hat{G}_\lambda = \text{Hol}(\mathcal{F}) \).
- \( \hat{G}_0 \) is the field \( (\mathcal{F}_x)_{x \in M} \).

Its \( C^* \)-algebra is isomorphic (via Fourier) to \( C_0(\mathcal{F}^*) \).
Quantization III: The $C^*$-algebra extension

As in the Lie groupoid case, restricting $\hat{G}$ to $[0, 1]$, we get an exact sequence:

$$0 \to C^*(M, \mathcal{F}) \otimes C_0((0, 1]) \to C^*(\hat{G}_{[0,1]}) \xrightarrow{ev_0} C_0(\mathcal{F}^*) \to 0$$

and for $t \in (0, 1]$ we have

$$ev_t : C^*(\hat{G}_{[0,1]}) \to C^*(M, \mathcal{F})$$

We get an asymptotic morphism

$$\alpha = \{\alpha_t = ev_t \circ (ev_0)^{-1} : C_0(\mathcal{F}^*) \to C^*(M, \mathcal{F})\}_{t \in (0,1]}$$

This quantizes the Lie-Poisson structure on $\mathcal{F}^*$.
Analysis I : Pseudodifferential Calculus

Pseudodifferential operators should be smoothing outside the 'units':

Start with a bi-submersion \((U, t, s) \in \mathcal{U}\) with an identity bisection, i.e. a submanifold \(V \subset U\) such that \(s, t\) coincide on \(V\) (and are local diffeomorphisms).

We also need:

- A \textbf{local isomorphism} \(\phi : U \to NV\) (\(NV = \text{normal bundle of } V\) in \(U\) - only defined near \(V\));
- A smooth \textbf{cut-off function} \(\chi\) on \(U\) s.t. \(\chi(u) = 1\) on \(V\) and \(\chi(u) = 0\) far from \(V\);
- A \textbf{symbol} \(a\) on \(\mathbb{N}^*\) (polyhomogeneous)
We can then make sense of an expression like

\[ k_\alpha(u) = \int \alpha(s(u), \xi) \exp(i\phi(u)\xi) \chi(u) d\xi, \]

**Theorem**

This way we get a multiplier of the \(*\)-algebra \( \mathcal{A}_U \).

**Proposition**

Modulo lower order, the operator \( k_\alpha \) only depends on the restriction of \( \alpha \) to \( \mathcal{F}^* \).

(Namely, if \( \alpha \) is of order \( m \) and vanishes on \( \mathcal{F}^* \), there exists \( b \) of order \( m - 1 \) such that \( k_\alpha \) and \( k_b \) define the same multiplier.)
Analysis III : The extension of order 0 pseudodifferential operators

We have the following facts:

- If $\alpha$ is of negative order, then $k_\alpha \in C^*(\mathcal{M},\mathcal{F})$;
- If $\alpha$ is of order 0, then $k_\alpha$ is bounded in every representation; it's a multiplier of $C^*(\mathcal{M},\mathcal{F})$;
- If $\alpha$ is of order 0 and its symbol vanishes outside $\mathcal{F}^*$, then $k_\alpha \in C^*(\mathcal{M},\mathcal{F})$.

We take the closure of the set of such operators and find an exact sequence of $C^*$-algebras:

$$0 \to C^*(\mathcal{M},\mathcal{F}) \to \Psi^*(\text{Hol}(\mathcal{F})) \to C_0(S\mathcal{F}^*) \to 0$$

We thus have an analytic index map $K^*(\mathcal{F}^*) \to K^*(C^*(\mathcal{M},\mathcal{F}))$. 
Geometry I: The nerve of an atlas

Given an atlas $\mathcal{U}$ of a foliation $(M, \mathcal{F})$ induces a groupoid-like simplicial structure. For $n \leq 1$ put

$$\mathcal{U}^{(n)} = \{(U_{i_1} \circ \ldots \circ U_{i_n}, t_{i_1 \ldots i_n}, s_{i_1 \ldots i_n})\}_{i_1 \ldots i_n \in I}$$

It is another atlas of bi-submersions for $\mathcal{F}$. Get simplicial maps $p_0, \ldots, p_n : \mathcal{U}^{(n)} \rightarrow \mathcal{U}^{(n-1)}$ where $p_0, p_n$ are projections and $p_i$ is defined by an adaptation map $\alpha_i : U_i \circ U_{i+1} \rightarrow U_i$. Namely,

$$p_i(u_1, \ldots, u_n) = (u_1, \ldots, u_{i-1}, \alpha_i(u_i, u_{i+1}), u_{i+2}, \ldots, u_n)$$
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Geometry II: Differential forms

For every $n \in \mathbb{N}$ denote $i_{[n]} = (i_1, \ldots, i_n)$ for a collection of $n$ indices in $I$. Now put $U_{i_{[n]}} = U_{i_1} \circ \ldots \circ U_{i_n}$ and let

$$\Omega^m(U^{(n)}) = \bigoplus_{i_1, \ldots, i_n \in I} C^\infty_c(U_{i_{[n]}}, \Omega^{1/2}(U_{i_{[n]}})) \otimes \Lambda^m T^*(U_{i_{[n]}})$$

**Definition**

Let $I^{n,m}$ be the subspace of $\Omega^m(U^n)$ spanned by the $p! (\theta)$ where $p : W \to U^{(n)}$ is a submersion and $\theta \in C^\infty_c(W; \Omega^{1/2}W)$ is such that there exists a morphism $q : W \to V$ of bi-submersions which is a submersion and $q!(\theta) = 0$.

Set

$$D_{U}^{n,m} = \Omega^m(U^{(n)})/I^{n,m}$$
Geometry III: the Čech-de Rham bicomplex

- The de Rham complex of $\mathcal{F}$ is

$$
\mathcal{A}_\mathcal{U} \xrightarrow{d} D^1_\mathcal{U} \xrightarrow{d} D^2_\mathcal{U} \xrightarrow{d} D^3_\mathcal{U} \xrightarrow{d} \ldots
$$

- Vertical differential is $\delta : D^{n,m}_\mathcal{U} \rightarrow D^{n+1,m}_\mathcal{U}$ defined by using any adaptation map $\alpha_j$ in the coordinates $(i_j, i_{j+1})$ and identity elsewhere. Namely, we put

$$
\delta(\omega)_{i_0 i_1 \ldots i_n} := \sum_{s=0}^{n+1} (-1)^s p_s^*(\omega),
$$

where

$$
p_s^*(\omega) = \begin{cases} 
p_0^*(\omega_{i_1 \ldots i_n}) & \text{if } s = 0 \\
p_s^*(\omega_{i_0 \ldots i_{s-1} k_s i_{s+2} \ldots i_n}) & \text{if } s = 1 \ldots n \\
p_{n+1}^*(\omega_{i_0 \ldots i_{n-1}}) & \text{if } s = n + 1
\end{cases}
$$