**1. Rieffel’s Deformation Quantization**

An interesting class of Poisson-structures comes from commuting vector fields.

If \( X_1, \ldots, X_n \) are mutually commuting vector fields, then for every skew symmetric matrix \( \theta^{ij} \in \mathfrak{g}(\mathbb{R}) \) the bivector field

\[
\theta = \frac{1}{2} \sum \theta^{ij} X_i \wedge X_j
\]

is a Poisson-tensor. Furthermore, if the flow of each vector field is complete, then

\[
a_t = f \circ \exp t \quad \text{defines an action on} \quad C^0(M).
\]

This is the point of departure for Rieffel’s “Strict deformation quantization for actions of \( \mathbb{R}^n \)” [RF].

If \( a \colon \mathbb{R}^n \times \mathbb{R} \rightarrow A \) is a strongly continuous, isometric action on a Fréchet-Algebra \( A \), then

\[
a \ast b = \frac{1}{2\pi} \int_a \omega(a, b) \mathrm{d}a
\]

defines a associative product on the subspace of smooth vectors \( A^\infty \) of \( A \).

If \( a \) is generated by the flows of commuting vector fields, the asymptotic expansion for \( h \to 0 \) yields the formal star product

\[
a \ast b = a \ast b + \frac{1}{h} \sum_{2 \leq n \leq \infty} a \ast (\lambda^n b) + \cdots
\]

i.e. the Weyl-Moyal product associated with the Poisson-tensor \( \theta \).

**2. The Quest for Positivity**

**Problem:** We are interested in states of the Rieffel deformed algebra \( A(h) \). If the undeformed algebra \( A \) is a classical observable algebra the states on \( A \) have a very concrete meaning. To every classical state we want to associate a certain quantum state on \( A(h) \).

The situation for the formal star products gives an idea how to approach this kind of association:

**Proposition 1.** Let \( g(a, b) = \theta(a, b) \) be the compatible metric corresponding to the Poisson-structure \( \theta \) and the almost complex structure \( J \). Then the operator

\[
S_\theta = \begin{pmatrix} \lambda & \theta \end{pmatrix} \in C^\infty(\Lambda^0) \quad \text{is an equivalence transformation for star products. Explicitly we have}
\]

\[
f \ast a \ast g = S_\theta^{-1} f \ast \theta \ast S_\theta^{-1} g
\]

By the properties of the Wick product it is clear that

\[
S_\theta \colon (A(h), \ast_{\omega_\wp}) \rightarrow (A(h), \ast_{\omega_\wp})
\]

is a positive map and that the positive elements of the Wick-algebra are positive elements of the undeformed algebra, so that

\[
\omega \circ S_\theta \text{ are states of the Weyl-algebra}
\]

for every classical state \( \omega \) [BW].

**Strategy:** We want to translate the operator \( S_\theta \) into the convergent setting and show that the nice properties of positiveness are conserved and are very similar to the formal case:

If \( (A(h), \ast_{\omega_\wp}) \) is a \( C^\ast \)-algebra, then there is a concrete construction of \( C^\ast \)-norms \( \| \cdot \|_{\ast_{\omega_\wp}} \) making \( (A(h), \ast_{\omega_\wp}) \) into a \( C^\ast \)-algebra. In this case we have

**Theorem 1 (Norm-continuity of \( S_\theta \)).** If \( (A(h), \ast_{\omega_\wp}) \) is a \( C^\ast \)-algebra and of \( (A(h), \ast_{\theta}) \) is the Rieffel deformed \( C^\ast \)-algebra, then

\[
S_\theta \colon A^\infty \rightarrow A
\]

is a positive element of the \( C^\ast \)-norms of \( A \) and \( A^\infty \).

The same holds for the restriction of \( S_\theta \) to \( A^\infty \rightarrow A^\infty \).

**4. Deformation of States in the \( C^\ast \)-Algebraic Setting**

For Rieffel’s construction \( \omega = \omega_\wp \) on the algebra \( A(h) = (A(h), \ast_\wp) \) forms a continuous field of \( C^\ast \)-algebras [RF, DX]. Continuous sections of the field are

\[
\Gamma = \left\{ \alpha \in \prod_h A(h) \mid \alpha \in A(h) \text{ is continuous} \right\}
\]

and \( (A(h), \Gamma_h) \) is a continuous-pred of \( C^\ast \)-algebras, i.e. \( (A(h), \Gamma_h) \) with

\[
\Gamma = \left\{ \alpha \in \prod_h A(h) \mid \alpha \in A(h) \text{ is continuous} \right\}
\]

is a continuous field of \( C^\ast \)-algebras.

Theorem 1 and 2 show that for every state \( \omega = \omega_\wp \) on the undeformed algebra \( A \)

\[
\omega_h = \omega \circ S_\theta \quad \text{is a state of the Rieffel deformed algebra} \quad A(h)
\]

We have shown, that for every continuous section \( \alpha \in \Gamma \) the map

\[
\Lambda \colon \alpha \rightarrow \omega_h(\alpha(h))
\]

is continuous. This means that

\[
\Lambda(\alpha) \in C^\ast(\mathbb{R}) \Rightarrow \text{form a continuous field of states over the continuous field} \quad (A(h), \Gamma)
\]

of \( C^\ast \)-algebras.